

# PRESENTATIONS OF AFFINE KAC-MOODY GROUPS

INNA CAPDEBOSCQ, KARINA KIRKINA, AND DMITRIY RUMYNIN

**ABSTRACT.** How many generators and relations does  $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$  need? In this paper we exhibit its explicit presentation with 9 generators and 44 relations.

We investigate presentations of affine Kac-Moody groups over finite fields. Our goal is to derive optimal finite presentations, independent of the field and with as few generators and relations as we can achieve. It turns out that any simply connected affine Kac-Moody group over a finite field has a presentation with at most 12 generators and 71 relations. We describe these presentations explicitly type by type. We also derive explicit presentations of Chevalley groups  $G(\mathbb{F}_q[t, t^{-1}])$  and explicit profinite presentations of profinite Chevalley groups  $G(\mathbb{F}_q[[t]])$ .

## 1. INTRODUCTION

Recently there has been a number of papers showing that various infinite families of groups have presentations with bounded number of generators and relations. In particular, the results of the following type were proved:

*Let  $\mathcal{A}$  be a certain family of groups. There exists  $C > 0$  such that for any group  $G \in \mathcal{A}$ ,  $G$  admits a presentation  $\sigma(G) = \langle D_{\sigma(G)} \mid R_{\sigma(G)} \rangle$  such that*

$$|D_{\sigma(G)}| + |R_{\sigma(G)}| < C.$$

This result is known if the family  $\mathcal{A}$  is a family of finite simple groups [GKKaL1, GKKaL2, GKKaL3], a family of affine Kac-Moody groups defined over finite fields [C1], and more recently a family of Chevalley groups over various rings [CLR]. In fact, Guralnick, Kantor, Kassabov and Lubotzky [GKKaL1, GKKaL3] provide a numerical bound on  $C$  in the case when  $\mathcal{A}$  is a family of finite simple groups (with the possible exception of  ${}^2G_2(3^a)$ ,  $a \geq 1$ ):

*If  $G$  is a non-sporadic quasisimple finite group (with the possible exception of  ${}^2G_2(3^a)$ ,  $a \geq 1$ ), then  $G$  has a presentation with at most 2 generators and 51 relations.*

The goal of the current paper is to provide a quantitative statement in the case when  $\mathcal{A}$  is a family of affine Kac-Moody groups over finite fields and use this to derive numerical bounds for some arithmetic groups defined over fields of positive characteristic.

**Theorem 1.1.** *Let  $G$  be a simply connected affine Kac-Moody group of rank  $n \geq 3$  defined over a finite field  $\mathbb{F}_q$ . If  $q \geq 4$ ,  $G$  has a presentation with 2 generators and at most 73 relations.*

*The result also holds if  $q \in \{2, 3\}$  provided that the Dynkin diagram of  $G$  is not of type  $\tilde{A}_2$  and does not contain a subdiagram of type  $B_2$  or  $G_2$  for  $q = 2$ , and of type  $G_2$  for  $q = 3$ . If  $G = \tilde{A}_2(2)$  or  $\tilde{A}_2(3)$ ,  $G$  has a presentation with at most 3 generators and 29 relations.*

Our results are not restricted to simply connected groups. Using our techniques we derive quantitative bounds on the presentations of arbitrary affine Kac-Moody groups over finite fields. These results can be found in Section 6.

The upper bound of 73 in Theorem 1.1 comes from the groups of type  $\tilde{C}_n^t$ . In other types the bounds are better, as stated in the next theorem.

**Theorem 1.2.** *Let  $G$  be a simply connected affine Kac-Moody group of rank  $n \geq 3$  defined over a finite field  $\mathbb{F}_q$ . If  $q \geq 4$ ,  $G$  has a presentation  $\sigma_G = \langle D_\sigma \mid R_\sigma \rangle$  where  $|D_\sigma|$  and  $|R_\sigma|$  are given*

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TABLE 1. Generators and Relations of  $\tilde{X}(q)$ 

Group	$ D_\sigma $	$ R_\sigma $	Group	$ D_\sigma $	$ R_\sigma $	$ D_\sigma $	$ R_\sigma $
				$q$ odd		$q$ even	
$\tilde{A}_2(q)$	5	26	$\tilde{B}_3(q)$	8	43	8	35
$\tilde{A}_n(q), 3 \leq n \leq 7$	7	35	$\tilde{B}_3^t(q)$	7	42	8	35
$\tilde{A}_n(q), n \geq 8$	9	43	$\tilde{B}_4(q), \tilde{B}_4^t(q)$	8	51	9	44
$\tilde{D}_4(q)$	7	38	$\tilde{B}_n(q), \tilde{B}_n^t(q), 5 \leq n \leq 8$	8	52	9	45
$\tilde{D}_5(q)$	7	39	$\tilde{B}_n(q), \tilde{B}_n(q)^t, n \geq 9$	9	56	10	49
$\tilde{D}_n(q), 6 \leq n \leq 8$	7	38	$\tilde{C}_2(q), \tilde{C}_2'(q)$	7	49	9	35
$\tilde{D}_n(q), n \geq 9$	8	42	$\tilde{C}_n(q), \tilde{C}_n'(q), n \geq 3$	10	69	12	55
$\tilde{E}_6(q)$	7	36	$\tilde{C}_2^t(q)$	8	50	9	35
$\tilde{E}_7(q)$	6	30	$\tilde{C}_n^t(q)$	12	71	12	55
$\tilde{E}_8(q)$	7	34	$\tilde{F}_4(q), \tilde{F}_4^t(q)$	8	50	9	43
$\tilde{G}_2(q), \tilde{G}_2^t(q)$	7	40					

in Table 1. The result also holds if  $q \in \{2, 3\}$  provided that the Dynkin diagram of  $G$  does not contain a subdiagram of type  $B_2$  or  $G_2$  for  $q = 2$ , and of type  $G_2$  for  $q = 3$ .

Many mathematicians encounter affine Kac-Moody groups defined over  $\mathbb{F}_q$  as Chevalley groups defined over  $\mathbb{F}_q[t, t^{-1}]$ :  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}]) \cong \tilde{\mathbf{G}}(\mathbb{F}_q)/Z$  where  $Z \cong \mathbb{F}_q^\times$  is a central subgroup of  $\tilde{\mathbf{G}}(\mathbb{F}_q)$  [MoRe, Section 2]. For example,  $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$  is the quotient of the simply connected affine Kac-Moody group  $\tilde{A}_{n-1}(q)$  by its central subgroup  $Z \cong \mathbb{F}_q^\times$ . Therefore we can obtain a presentation of  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$  from a presentation of  $\tilde{\mathbf{G}}(\mathbb{F}_q)$  (as in Table 1) by adding one extra relation to kill a generator of  $Z$ .

The groups  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$  can be generated by two elements (cf. Theorem 2.7). Therefore we can change our presentation to a presentation of  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$  in these two generators. This change of generators costs two extra relations (cf. Lemma 2.1). The next theorem summarises this.

**Theorem 1.3.** *Let  $\mathbf{G}$  be a simple simply connected Chevalley group scheme of rank  $n \geq 2$ . Take  $q = p^a$ ,  $a \geq 1$  with  $p$  a prime and set  $G = \mathbf{G}(\mathbb{F}_q[t, t^{-1}])$ . Then  $G$  has a presentation with 2 generators and at most 72 relations with the possible exceptions of  $A_2(\mathbb{F}_2[t, t^{-1}])$ ,  $B_n(\mathbb{F}_2[t, t^{-1}])$ ,  $C_n(\mathbb{F}_2[t, t^{-1}])$ ,  $G_2(\mathbb{F}_2[t, t^{-1}])$ ,  $F_4(\mathbb{F}_2[t, t^{-1}])$ ,  $A_2(\mathbb{F}_3[t, t^{-1}])$  and  $G_2(\mathbb{F}_3[t, t^{-1}])$ . If  $G = A_2(\mathbb{F}_2[t, t^{-1}])$  or  $A_2(\mathbb{F}_3[t, t^{-1}])$ ,  $G$  has a presentation with at most 3 generators and 30 relations.*

The precise number of generators and relations in a presentation of  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$  can be deduced from Table 1 by adding 1 relation, and for a presentation with 2 generators can be found in Table 2.

Capdeboscq, Lubotzky and Remy connect the presentations of Chevalley groups over  $\mathbb{F}_q[t, t^{-1}]$  with the profinite presentations of Chevalley groups defined over  $\mathbb{F}_q[[t]]$  [CLR, Proposition 1.2]. An immediate consequence of their Proposition 1.2 combined with our Theorem 1.3, is the following statement.

**Theorem 1.4.** *Let  $\mathbf{G}$  be a simple simply connected Chevalley group scheme of rank at least 2. For  $q = p^a$ ,  $a \geq 1$ ,  $p$  a prime, consider a profinite group  $G = \mathbf{G}(\mathbb{F}_q[[t]])$ . Then  $G$  has a profinite presentation with 2 generators and at most 73 relations with the possible exceptions of  $A_2(\mathbb{F}_2[[t]])$ ,  $B_n(\mathbb{F}_2[[t]])$ ,  $C_n(\mathbb{F}_2[[t]])$ ,  $G_2(\mathbb{F}_2[[t]])$ ,  $F_4(\mathbb{F}_2[[t]])$ ,  $A_2(\mathbb{F}_3[[t]])$  and  $G_2(\mathbb{F}_3[[t]])$ . If  $G = A_2(\mathbb{F}_2[[t]])$  or  $A_2(\mathbb{F}_3[[t]])$ , then  $G$  has a profinite presentation with at most 3 generators and 31 relations.*

TABLE 2. Relations of  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$  with 2 generators

type	$\mathbf{G}$	$ R_\sigma $	type	$\mathbf{G}$	$ R_\sigma , q \text{ odd}$	$ R_\sigma , q \text{ even}$
$A_{n-1}$	$\text{SL}_3$	29	$B_n$	$\text{Spin}_7$	45	38
	$\text{SL}_n, 4 \leq n \leq 8$	38		$\text{Spin}_9$	54	47
	$\text{SL}_n, n \geq 9$	46		$\text{Spin}_{2n+1}, 5 \leq n \leq 8$	55	48
$D_n$	$\text{Spin}_8$ or $\text{Spin}_{2n}, 6 \leq n \leq 8$	41		$\text{Spin}_{2n+1}, n \geq 9$	59	52
	$\text{Spin}_{10}$	42				
	$\text{Spin}_{2n}, n \geq 9$	45	$C_n$	$\text{Sp}_4$	52	38
$E_n$	$E_6$	39		$\text{Sp}_{2n}, n \geq 3$	72	58
	$E_7$	33	$F_4$	$F_4$	53	46
	$E_8$	37				
$G_2$	$G_2$	43				

Let us describe the structure of the paper. In Section 2 we outline the proof providing background results. We introduce simply connected Kac-Moody groups and show that a Kac-Moody subgroup of a simply connected Kac-Moody group is a simply connected Kac-Moody group itself. In Section 3 we examine the presentation of a finite group in type  $D_n$ . In Section 4 we analyse the presentations of all untwisted Kac-Moody groups giving details on the case-by-case basis. We examine the twisted Kac-Moody groups in Section 5. In Section 6 we deal with the presentations of adjoint Kac-Moody groups and classical groups over  $\mathbb{F}_q[t, t^{-1}]$ . We record all our findings in Tables 4 and 5.

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## 2. BACKGROUND RESULTS AND OUTLINE OF THE PROOF

**2.1. Presentations of finite quasisimple groups.** Let us recall one of the main results of [GKKaL3]. It asserts that every finite quasisimple group of Lie type admits a presentation with 2 generators and at most 51 relations. The authors achieve this in two steps. First they give a presentation for each family of groups of Lie type with very few generators and relations. Then, since every finite quasisimple group is 2-generated (see [AsG, Theorem B]), they use a reduction lemma (Lemma 2.3 of [GKKaL3]) to obtain the main result. We now restate this reduction lemma to include the case of infinite groups.

**Lemma 2.1** (Lemma 2.3 of [GKKaL3]). *Let  $\sigma = \langle X \mid R \rangle$  be a finite presentation of a group  $G$ ,  $\pi : F\langle X \rangle \rightarrow G$  the corresponding natural map from a free group. If  $D$  is a finite subset of  $G$  such that  $G = \langle D \rangle$ , then  $G$  also has a presentation  $\langle D \mid R' \rangle$  such that  $|R'| = |D| + |R| - |\pi(X) \cap D|$ .*

*Proof.* Let  $D = \{d_1, d_2, \dots, d_l\}$ . The new presentation is obtained by a sequence of Tietze transformations. First we add generators:

$$\sigma' = \langle X \cup D \mid R \cup \{d_i = \delta_i(X) \mid d_i \in D \setminus \pi(X)\} \cup \{d_j = x_j^* \mid d_j \in D \cap \pi(X)\} \rangle$$

where the second set in the union contains one relation for each  $d_i \in D \setminus \pi(X)$  expressing  $d_i$  as a word  $\delta_i(X)$  in  $X$ , while the third set in the union contains one relation for each  $d_j \in D \cap \pi(X)$  where  $d_j = \pi(x_j^*)$  for some  $x_j^* \in X$ . We then remove  $X$ :

$$\sigma'' = \langle D \mid R|_{x_j = \chi_j(D)} \cup \{d_i = \delta_i(\chi_1(D), \dots, \chi_k(D)) \text{ for } d_i \in D \setminus \pi(X)\} \rangle$$

TABLE 3. Presentations of  $\mathbf{G}(\mathbb{F}_q)$  [GKKaL3]

Group	generators	relations	label
$\mathrm{SL}(2, q)$	3	9	$\sigma_1$
$\mathrm{SL}(3, q)$	4	14	$\sigma_2$
$\mathrm{SL}(4, q)$	5 (6)	20 (22)	$\sigma_3$ ( $\sigma_4$ )
$\mathrm{SL}(n, q)$ , $5 \leq n \leq 8$	5 (6)	21 (22)	$\sigma_5$ ( $\sigma_6$ )
$\mathrm{SL}(n, q)$ , $n \geq 9$	6 (7)	25 (26)	$\sigma_7$ ( $\sigma_8$ )
$\mathrm{Sp}(4, q)$ , $q$ odd	5 (6)	27 (28)	$\sigma_9$ ( $\sigma_{18}$ )
$\mathrm{Sp}(4, q)$ , $q$ even	6	20	$\sigma_{10}$
$\mathrm{Sp}(2n, q)$ , $n \geq 3$ , $q$ odd	8	47	$\sigma_{11}$
$\mathrm{Sp}(2n, q)$ , $n \geq 3$ , $q$ even	9	40	$\sigma_{12}$
$\mathrm{Spin}(2n+1, q)$ , $n \geq 3$ , $q$ odd	9	48	$\sigma_{13}$
$\mathrm{Spin}(8, q)$	6	29	$\sigma_{14}$
$\mathrm{Spin}(2n, q)$ , $5 \leq n \leq 8$	6	30	$\sigma_{15}$
$\mathrm{Spin}(2n, q)$ , $n \geq 9$	7	34	$\sigma_{16}$
$\mathrm{G}_2(q)$	6	31	$\sigma_{17}$

where each  $x_j \in X$  is expressed as a word  $\chi_j(D)$  in  $D$  and  $R|_{x_j=\chi_j(D)}$  is a result of this substitution in the relations  $R$ .  $\square$

This result implies that, if a 2-generated group has a presentation with  $n$  generators and  $m$  relations, it also has a presentation with 2 generators and  $m+2$  relations.

We need some of the presentations of quasisimple groups of Lie type obtained in [GKKaL3]. We exhibit them in Table 3 (cf. Table 1 of [GKKaL3, p. 93]).

**Definition 2.2.** *Let  $B$  be a group and  $A$  its subgroup. Suppose further that  $B$  has a finite presentation  $\sigma_B = \langle X_B \mid R_B \rangle$  and  $A$  has a presentation  $\sigma_A = \langle X_A \mid R_A \rangle$  such that  $X_A \subset X_B$  and  $R_A \subseteq R_B$ . Then we say that*

$$\sigma_A \subseteq \sigma_B.$$

Let us use this definition to explain the connection between presentations of quasisimple groups of Lie type in Table 3. Theorem 4.5. of [GKKaL3] gives a presentation  $\sigma_1$  of  $\mathrm{SL}(2, q)$ . Consider a group  $G = \mathrm{SL}(3, q)$ . If  $\{\alpha_1, \alpha_2\}$  is the set of simple roots of  $\mathrm{SL}(3, q)$ , then  $G$  contains a subgroup  $L = \langle X_{\alpha_1}, X_{-\alpha_1} \rangle \cong \mathrm{SL}(2, q)$ , where  $X_{\alpha_1}$  and  $X_{-\alpha_1}$  are the root subgroups of  $\mathrm{SL}(3, q)$  (cf. [Car2]). Theorem 5.1 of [GKKaL3] gives a presentation  $\sigma_2$  of  $\mathrm{SL}(3, q)$  that contains  $\sigma_L = \sigma_1$ .

Theorem 6.1 of [GKKaL3] gives presentations  $\sigma_4$  of  $\mathrm{SL}(4, q)$ ,  $\sigma_6$  of  $\mathrm{SL}(n, q)$  for  $5 \leq n \leq 8$ , and  $\sigma_8$  of  $\mathrm{SL}(n, q)$  for  $n \geq 9$ . If  $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$  is the set of simple roots of  $\mathrm{SL}(n, q)$ , the group  $\mathrm{SL}(n, q)$  contains a subgroup  $M = \langle X_{\alpha_1}, X_{-\alpha_1}, X_{\alpha_2}, X_{-\alpha_2} \rangle \cong \mathrm{SL}_3(q)$ . From the proof of Theorem 6.1 it follows that  $\sigma_2 \subseteq \sigma_i$  for  $i = 4, 6$  and  $8$  where  $\sigma_2$  is the presentation of  $M$ . Thus we obtain that

$$\sigma_1 \subseteq \sigma_2 \subseteq \sigma_i \text{ for } i = 4, 6, 8.$$

Theorem 7.1 and Remark 7.4 of [GKKaL3] give presentations  $\sigma_9$ ,  $\sigma_{18}$  and  $\sigma_{10}$  of  $\mathrm{Sp}(4, q)$  ( $\sigma_9$ ,  $\sigma_{18}$  in the case when  $q$  is odd, and  $\sigma_{10}$  in the case when  $q$  is even). Let  $\alpha_2$  be a short root,  $\alpha_1$  a long root. All three presentations contain a presentation  $\sigma_1$  of its short-root subgroup  $L_2 = \langle X_{\alpha_2}, X_{-\alpha_2} \rangle \cong \mathrm{SL}(2, q)$ . Besides  $\sigma_{18}$  contains a presentation  $\sigma_1$  of its long-root subgroup  $L_1 = \langle X_{\alpha_1}, X_{-\alpha_1} \rangle$ :

$$\sigma_1(L_1) \subseteq \sigma_{18}, \quad \sigma_1(L_2) \subseteq \sigma_i \text{ for } i = 9, 10, 18.$$

Finally, we also need a presentation of family of groups  $\text{Spin}(2n, q)$ ,  $n \geq 4$ ,  $q = p^a$ . The result of [GKKaL3] says that such groups have presentations with 9 generators and 42 relations. We shorten this estimate to 7 generators and 34 relations in Section 3.

**2.2. Simply connected Kac-Moody groups and their presentations.** Recall that Kac-Moody groups over arbitrary fields were defined by Tits [T]. We are only interested in the case when the group is split and the field of the definition  $\mathbb{F} = \mathbb{F}_q$  is a finite field of  $q = p^a$  elements ( $a \geq 1$  and  $p$  a prime).

Let  $A = (A_{ij})_{n \times n}$  be a generalised Cartan matrix,  $\mathfrak{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$  a root datum of type  $A$ . Recall that this means

- $I = \{1, 2, \dots, n\}$ ,
- $\mathcal{Y}$  is a free finitely generated abelian group,
- $\mathcal{X} = \mathcal{Y}^* = \text{hom}(\mathcal{Y}, \mathbb{Z})$  is its dual group,
- $\Pi = \{\alpha_1, \dots, \alpha_n\}$  is a set of simple roots, where  $\alpha_i \in \mathcal{X}$ ,
- $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$  is a set of simple coroots, where  $\alpha_i^\vee \in \mathcal{Y}$ ,
- for all  $i, j \in I$ ,  $\alpha_i(\alpha_j^\vee) = A_{ij}$ .

From Tits' definition an explicit presentation of these groups was derived by Carter, a presentation by the explicit set of generators and relations à la Steinberg (cf. [Car1] p. 224). The presentation depends on the field and root datum, so the resulting group can be denoted  $G_{\mathfrak{D}}(\mathbb{F})$ . If  $A$  is not of finite type (i.e.,  $A$  is not a Cartan matrix from classical Lie theory), the standard presentation is infinite: the number of generators and the number of relations are both infinite.

Recall that  $G_{\mathfrak{D}}$  is *simply connected* if  $\Pi^\vee$  is a basis of  $\mathcal{Y}$  (see Section 6 for an example of a simply connected affine group), and  $G_{\mathfrak{D}}(\mathbb{F})$  is *2-spherical* if  $A$  is 2-spherical. The latter means that for each  $J \subseteq I$  with  $|J| = 2$ , the submatrix  $A_J := (A_{ij})_{i,j \in J}$  is a classical Cartan matrix. Now, for each  $\alpha \in \Pi \cup -\Pi$ , let  $X_\alpha$  be a root subgroup of  $G_{\mathfrak{D}}(\mathbb{F})$ . Then  $X_\alpha \cong (\mathbb{F}_q, +)$ , and for all  $i, j \in I$  with  $i \neq j$ , set

$$L_i := \langle X_{\alpha_i} \cup X_{-\alpha_i} \rangle \quad \text{and} \quad L_{ij} := \langle L_i \cup L_j \rangle = L_{ji}.$$

Abramenko and Muhlherr [AM] (see also [Ca, Th. 3.7]) proved a significant new theorem about the presentations of a large class of Kac-Moody groups. In particular, they showed that those groups were finitely presented.

**Theorem.** (Abramenko, Muhlherr) *Let  $A$  be a 2-spherical generalised Cartan matrix and  $\mathfrak{D}$  a simply-connected root datum corresponding to  $A$ . Suppose that the field  $\mathbb{F}$  is finite and the following condition holds:*

$$L_{ij}/Z(L_{ij}) \not\cong B_2(2), G_2(2), G_2(3), {}^2F_4(2) \quad \text{for all } i, j \in I \quad (*).$$

*Let  $\tilde{G}$  be the direct limit of the inductive system formed by the  $L_i$  and  $L_{ij}$  for  $i, j \in I$ , with the natural inclusions. Then the canonical homomorphism  $\tilde{G} \rightarrow G_{\mathfrak{D}}(\mathbb{F})$  is an isomorphism.*

For an insightful description of 2-spherical Kac-Moody groups we refer the reader to the paper of Caprace [Ca].

The following observation is useful.

**Proposition 2.3.** *Let  $A$  be a 2-spherical generalised Cartan matrix and  $\mathfrak{D}$  a simply connected root datum corresponding to  $A$ . Suppose that the field  $\mathbb{F}$  is finite and condition  $(*)$  holds. Let  $J \subseteq I$  and*

$$L_J = \langle L_i \mid i \in J \rangle.$$

*Then  $L_J$  is a simply connected Kac-Moody group  $G_{\mathfrak{D}(J)}(\mathbb{F})$  with a root datum of type  $A_J = (A_{ij})_{i,j \in J}$ .*

*Proof.* It is clear once you construct the root datum for the  $L_J$ . If  $J \subseteq I$ , let  $\Theta = \{\alpha_i \mid i \in J\}$ . Then  $\Theta^\vee = \{\alpha_i^\vee \mid i \in J\}$  is the corresponding subset of  $\Pi^\vee$ . Since  $\mathfrak{D}$  is simply connected, the coroot lattice splits into a direct sum

$$Y = \langle \Theta^\vee \rangle \oplus \langle \Pi^\vee \setminus \Theta^\vee \rangle.$$

Hence, the root datum for  $L_J$  is

$$\mathfrak{D}(J) = (J, A_J, X/\langle \Theta^\vee \rangle^\perp, \langle \Theta^\vee \rangle, \Theta, \Theta^\vee).$$

Thus,  $\mathfrak{D}(J)$  is simply connected.  $\square$

A direct consequence of the result of Abramenko and Muhlherr is that  $G_{\mathfrak{D}}(\mathbb{F}_q)$  has a presentation whose generators are generators of root subgroups  $X_i$ 's,  $i \in I$  and with the relations that are defining relations of all  $L_i$ 's and  $L_{ij}$ 's with  $i, j \in I, i \neq j$ . This allowed Capdeboscq [C1, Theorem 2.1] to show that the affine Kac-Moody groups (of rank at least 3) defined over finite fields had bounded presentations.

**Theorem.** (Capdeboscq) *There exists  $C > 0$  such that if  $\tilde{X}(q)$  is an affine Kac-Moody group of rank at least 3 defined over a field  $\mathbb{F}_q$  with  $q = p^a \geq 4$  ( $p$  a prime), then  $\tilde{X}(q)$  has a presentation  $\sigma_{\tilde{X}(q)} = \langle D_{\tilde{X}(q)} \mid R_{\tilde{X}(q)} \rangle$  such that*

$$|D_{\tilde{X}(q)}| + |R_{\tilde{X}(q)}| < C.$$

This result is based on the following observation (see the proof of Theorem 2.1 of [C1]) which is a direct consequence of the result of Abramenko and Muhlherr.

**Proposition 2.4.** *Let  $\tilde{X}(q)$  be a simply connected 2-spherical split Kac-Moody group of rank  $n \geq 3$  over the field of  $q$  elements. Let  $\Delta$  be the Dynkin diagram of  $\tilde{X}(q)$  with the vertices labelled by  $\beta_1, \dots, \beta_n$ . Suppose further that if  $\Delta$  contains a subdiagram of type  $B_2$ , then  $q \geq 3$ , and if it contains a subdiagram of type  $G_2$ ,  $q \geq 4$ .*

*Suppose that  $\Delta$  contains  $k$  proper subdiagrams  $\Delta_1, \Delta_2, \dots, \Delta_k$  such that  $\Delta = \cup_{i=1}^k \Delta_i$ , and each pair of vertices of  $\Delta$  is contained in  $\Delta_i$  for some  $i \in \{1, \dots, k\}$ . Let  $X_i(q) := \langle L_j \mid \alpha_j \in \Delta_i \rangle$ . If  $\sigma_{X_i(q)} = \langle D_i \mid R_i \rangle$  is a presentation of  $X_i(q)$ , then  $\tilde{X}(q)$  has a presentation*

$$\sigma_{\tilde{X}(q)} = \langle D_1 \cup D_2 \cup \dots \cup D_k \mid R_1 \cup \dots \cup R_k \cup \bigcup_{1 \leq i < j \leq k} R_{ij} \rangle$$

where  $R_{ij}$  are the relations coming from identifying the generators of  $X_{ij}(q) := \langle L_k \mid \alpha_k \in \Delta_i \cap \Delta_j \rangle$  in  $X_i(q)$  and in  $X_j(q)$ .

Suppose now that  $\Delta = \Delta_1 \cup \Delta_2$  on the level of vertices, that is,  $\Delta$  may contain edges, not included in  $\Delta_1$  or  $\Delta_2$ . We cannot use Proposition 2.4 immediately because it is not true that any pair of vertices is contained in  $\Delta_i$ . To remedy it we may introduce  $\Delta_3$  into the picture. Let  $\Delta_3$  be a subdiagram of  $\Delta$  based on vertices of  $(\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1)$ : now any pair of vertices of  $\Delta$  is contained in some  $\Delta_i$ . Proposition 2.4 gives a presentation

$$\sigma_{\tilde{X}(q)} = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Let  $X_3^1(q)$  be the subgroup of  $\tilde{X}(q)$  corresponding to  $\Delta_1 \setminus \Delta_2$ , and  $\sigma_{X_3^1(q)} = \langle D_3^1 \mid R_3^1 \rangle$  its presentation. *Mutatis mutandis*,  $\sigma_{X_3^2(q)} = \langle D_3^2 \mid R_3^2 \rangle$  for  $\Delta_2 \setminus \Delta_1$ . Now we can choose a special presentation of  $X_3(q)$ :

$$\sigma_{X_3(q)} = \langle D_3^1 \cup D_3^2 \mid R_3^1 \cup R_3^2 \cup R_3^{**} \rangle$$

where  $R_3^{**}$  are those relations that include generators in both  $D_3^1$  and  $D_3^2$ . Using  $\sigma_{X_3(q)}$  in  $\sigma_{\tilde{X}(q)}$  allows us to eliminate generators in  $D_3$  with a single Tietze transformation. Since  $\Delta_1 \supseteq \Delta_1 \setminus \Delta_2$  and  $\Delta_2 \supseteq \Delta_2 \setminus \Delta_1$ , many relations become superfluous as summarised in the next corollary.

**Corollary 2.5.** *Let  $\tilde{X}(q)$  be a simply connected 2-spherical split Kac-Moody group of rank  $n \geq 3$  over the field of  $q$  elements. Let  $\Delta$  be the Dynkin diagram of  $\tilde{X}(q)$  with the vertices labelled by  $\beta_1, \dots, \beta_n$ . Suppose further that if  $\Delta$  contains a subdiagram of type  $B_2$ , then  $q \geq 3$ , and if it contains a subdiagram of type  $G_2$ ,  $q \geq 4$ .*

*Suppose that  $\Delta$  contains three proper subdiagrams  $\Delta_1, \Delta_2$  and  $\Delta_3$  such that  $\beta_1, \dots, \beta_n \in \Delta_1 \cup \Delta_2$  and  $\Delta_3$  is a subdiagram of  $\Delta$  based on vertices of  $(\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1)$ . If for  $i = 1, 2$ ,  $\sigma_{X_i(q)} =$*

$\langle D_i \mid R_i \rangle$  is a presentation of  $X_i(q)$  and  $\sigma_{X_3(q)}$  is as described before this corollary, then  $\tilde{X}(q)$  has a presentation

$$\sigma_{\tilde{X}} = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

where the relations in  $R_3^*$  are obtained from the relations in  $R_3^{**}$  by substituting generators in  $D_3$  with their expressions via generators of  $D_1$  or  $D_2$ .

**2.3. 2-Generation.** A non-abelian finite simple group  $G$  can be generated by two elements [AsG, Theorem B]. We are interested in  $h(G)$ , the largest non-negative integer such that  $G^{h(G)}$  can be generated by two elements. Maróti and Tamburini have proved that  $h(G) > 2\sqrt{|G|}$  [MT, Theorem 1.1].

Consider a finite group  $H = G_1^{m_1} \times \dots \times G_t^{m_t}$  where each  $G_i$  is a non-abelian simple group. It is known that a subset of  $H$  generates  $H$  if and only if its projection into  $G_i^{m_i}$  generates  $G_i^{m_i}$  for each  $i$  [KL, Lemma 5]. We combine all this information in the following statement.

**Proposition 2.6.** *Let  $G = G_1^a \times G_2^b$  where  $G_1$  and  $G_2$  are finite non-abelian quasisimple groups and  $a$  and  $b$  are non-negative integers with  $0 \leq a, b \leq 3$ . Then  $G$  is 2-generated.*

*Proof.* The smallest order of a finite non-abelian simple group is 60. Hence, for any finite simple group  $h(G) \geq 16$ . The result now follows immediately from the combination of [AsG, Theorem B], [MT, Theorem 1.1] and [KL, Lemma 5] and the fact that a quasisimple finite group is a Frattini cover of a finite simple group.  $\square$

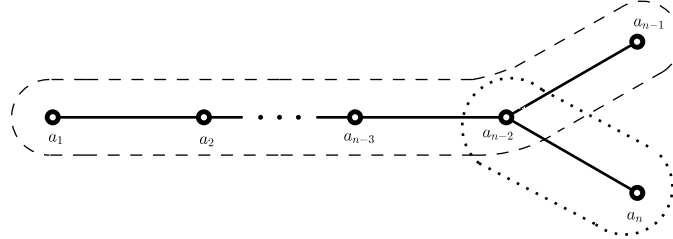
Now suppose that  $\tilde{X}(q)$  is a Kac-Moody group over the field  $\mathbb{F}_q$  (where  $q = p^a$ ,  $a \geq 1$ ,  $p$  a prime). We will need the following result proved in [C2] for large enough  $q$  and clarified in [CR] to include the small values of  $p$  and  $q$ .

**Theorem 2.7.** *Let  $\tilde{X}(q)$  be a simply connected affine Kac-Moody group of rank  $n \geq 3$  defined over a finite field  $\mathbb{F}_q$ . Then  $\tilde{X}(q)$  is generated by 2 elements, with the possible exceptions of  $\tilde{A}_2(2)$  and  $\tilde{A}_2(3)$  in which case it is generated by at most 3 elements.*

2.4. In the next three section we are going to use Proposition 2.4 and its corollary to obtain bounded presentations of Kac-Moody groups over finite fields. We do this by looking at affine groups case-by-case. We then record our findings in Table 1. There are 16 types of affine Dynkin diagrams that are listed in the next section. We use the Dynkin labels for affine diagrams [Car2].

### 3. PRESENTATION OF $\text{Spin}(2n, q)$

To obtain a presentation of  $G = \text{Spin}(2n, q)$ ,  $n \geq 4$ , we use Corollary 2.5.



By Proposition 2.3, the groups  $X_1(q)$ ,  $X_2(q)$  and  $X_3(q)$  are simply connected. Let  $\Delta_1$  be the subdiagram of  $\Delta$  whose vertices are the  $n-1$  nodes  $a_1, \dots, a_{n-1}$ . It has type  $A_{n-1}$  and so  $X_1(q) \cong \text{SL}(n, q)$ . Let  $\Delta_2$  be the subdiagram of  $\Delta$  whose vertices are the nodes  $a_{n-2}$  and  $a_n$ . It is of type  $A_2$  and so  $X_2(q) \cong \text{SL}(3, q)$ . Then  $\Delta_3$  is the subdiagram of  $\Delta$  based on all vertices but  $a_{n-2}$ , thus of type  $A_{n-3} \times A_1 \times A_1$ . Hence,  $X_3(q) \cong \text{SL}(n-2, q) \times \text{SL}(2, q) \times \text{SL}(2, q)$ . Clearly,  $\Delta = \cup_{i=1}^3 \Delta_i$ . Therefore  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

as described in Corollary 2.5.

Take a presentation  $\sigma_{X_1(q)}$ . If  $n \geq 9$ ,  $\sigma_{X_1(q)} = \sigma_7$ , if  $5 \leq n \leq 8$ ,  $\sigma_{X_1(q)} = \sigma_5$ , and if  $n = 4$ ,  $\sigma_{X_1(q)} = \sigma_3$ . Consider a subgroup  $X = L_{n-2}$  of  $G$ . Its Dynkin diagram is of type  $A_1$  and so by Proposition 2.3,  $X \cong \text{SL}(2, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$  with  $|D_X| = 3$  and  $|R_X| = 9$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ . The group  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_2$  with  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ , obviously,  $D_X \subseteq X_1(q)$ . Thus elements of  $D_X$  can be expressed in terms of elements of  $D_1$ . Moreover, the relations  $R_X$  hold, as they hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

If  $n \geq 9$ , we use  $\sigma_7$  that requires 6 generators and 25 relations, so that  $|D_1 \cup (D_2 \setminus D_X)| = 6 + 4 - 3 = 7$  and  $|R_1 \cup (R_2 \setminus R_X)| = 25 + 14 - 9 = 30$ .

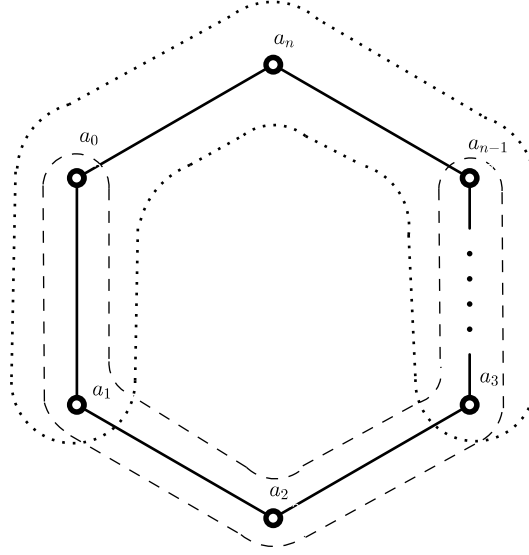
Consider  $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q))$  where  $X_3(q) \cap X_1(q) \cong \text{SL}(n-2, q) \times \text{SL}(2, q)$  and  $X_3(q) \cap X_2(q) \cong \text{SL}(2, q)$ . Each factor has two generators (Proposition 2.6). Denote them by  $a_1, b_1$  and  $a_2, b_2$  correspondingly. Then  $R_3^* = \{[a_1, a_2] = [b_1, b_2] = [a_1, b_2] = [a_2, b_1] = 1\}$  and  $|R_3^*| = 4$ .

Therefore if  $n \geq 9$ ,  $G$  has a presentation with 7 generators and 34 relations. By a similar calculation, if  $n = 4$ ,  $G$  has 6 generators and 29 relations, and if  $5 \leq n \leq 8$ ,  $G$  has 6 generators and 30 relations.

#### 4. UNTWISTED AFFINE KAC-MOODY GROUPS

We now go through the calculations for the 9 types of affine untwisted simply connected Kac-Moody groups.

4.1.  $\tilde{A}_n(q)$ ,  $n \geq 4$ . The Dynkin diagram  $\Delta$  of  $G = \tilde{A}_n(q)$  consists of  $n + 1$  vertices:



We now use Corollary 2.5. Notice, that by Proposition 2.3, the groups  $X_1(q)$ ,  $X_2(q)$  and  $X_3(q)$  are simply connected. Let  $\Delta_1$  be the subdiagram of  $\Delta$  whose vertices are the  $n$  nodes  $a_0, a_1, a_2, \dots, a_{n-1}$ . It has type  $A_n$  and so  $X_1(q) \cong \text{SL}(n+1, q)$ . Let  $\Delta_2$  be the subdiagram of  $\Delta$  whose vertices are the  $n$  nodes  $a_1, a_0, a_n, a_{n-1}, \dots, a_3$ . It is of type  $A_n$  and so  $X_2(q) \cong \text{SL}(n+1, q)$ . Then  $\Delta_3$  is the subdiagram of  $\Delta$  whose vertices are the two nodes  $a_2$  and  $a_n$ . This diagram is of type



$A_1 \times A_1$  and thus corresponds to a subgroup  $X_3(q) \cong \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$ . Clearly,  $\Delta = \cup_{i=1}^2 \Delta_i$ . Therefore  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle$$

as described in Corollary 2.5.

Let us find presentations of  $X_i(q)$  for  $i = 1, 2, 3$ . Consider a subgroup  $X$  of  $G$  that is generated by  $L_0$  and  $L_1$ . Its Dynkin diagram is of type  $A_2$  and so by Proposition 2.3,  $X \cong \mathrm{SL}(3, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_2$  with 4 generators and 14 relations. Now  $X \leq X_i(q)$  for  $i = 1, 2$ . Moreover,  $X_1(q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_7$  that requires 6 generators and 25 relations. Now  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_8$  with  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and relations  $R_X$  already hold. We use Tietze's transformations to eliminate  $D_X$  and  $R_X$ :

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \cup R_{12} \rangle.$$

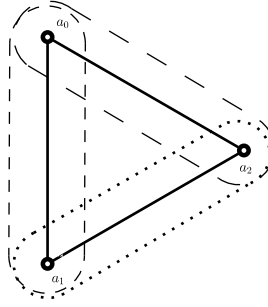
Now  $\sigma_8$  requires 7 generators and 26 relations, and hence  $|D_1 \cup (D_2 \setminus D_X)| = 6 + 7 - 4 = 9$  and  $|R_1 \cup (R_2 \setminus R_X)| = 25 + 26 - 14 = 37$ .

Consider  $X_3(q) = L_2 \times L_n$ . Since  $L_2 \cong L_n \cong \mathrm{SL}(2, q)$ , each factor has 2 generators (Proposition 2.6). Denote them by  $a_2, b_2$  and  $a_n, b_n$  correspondingly. Then  $R_3^* := \{[a_1, a_2] = [b_1, b_2] = [a_1, b_2] = [a_2, b_1] = 1\}$  and so  $|R_3^*| = 4$ .

Finally,  $\Delta_1 \cap \Delta_2$  is of type  $A_2 \times A_{n-3}$ . The corresponding group  $\mathrm{SL}(3, q) \times \mathrm{SL}(n-2, q)$  has 2 generators (Proposition 2.6). We call them  $c_1, d_1$  as elements of  $X_1(q)$  and  $c_2, d_2$  as elements of  $X_2(q)$ . Then  $R_{12} = \{c_1 = c_2, d_1 = d_2\}$  and so  $|R_{12}| = 2$ .

It follows that  $G$  has a presentation with 9 generators and  $37 + 4 + 2 = 43$  relations. If  $4 \leq n \leq 7$ , we may use  $\sigma_5$  and  $\sigma_6$  instead of  $\sigma_7$  and  $\sigma_8$ . Then  $G$  has a presentation with 7 generators and 35 relations.

4.2.  $\tilde{A}_2(q)$ . To obtain a presentation of  $G = \tilde{A}_2(q)$ , we start with Proposition 2.4.



Take  $\Delta_1 = A_2$  based on  $a_0$  and  $a_1$ ,  $\Delta_2 = A_2$  based on  $a_0$  and  $a_2$ , and  $\Delta_3 = A_2$  based on  $a_1$  and  $a_2$ . Clearly,  $\Delta = \cup_{i=1}^3 \Delta_i$  and every pair of vertices is contained in at least one of  $\Delta_i$ . Thus  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Since  $X_i(q) \cong A_2(q)$ ,  $X_i(q) \cong \mathrm{SL}(3, q)$  by Proposition 2.3, and  $X_i(q)$  has a presentation  $\sigma_{X_i(q)} = \sigma_2$  ( $i = 1, 2, 3$ ). Take  $X = L_0$ . Then for  $i = 1, 2$ ,  $X \leq X_i(q)$  and  $\sigma_X \subseteq \sigma_{X_i(q)}$ . It follows that  $|D_1 \cup D_2| = 4 + 4 - 3 = 5$ ,  $|R_1 \cup R_2| = 14 + 14 - 9 = 19$  and  $R_{12} \subseteq R_1 \cup R_2$ .

Take now  $Y = L_1$ . Without loss of generality we may assume that  $\sigma_Y \subseteq \sigma_{X_3(q)}$  with  $\sigma_Y = \sigma_1$ . Then by Theorem 5.1 of [GKKaL3],  $\sigma_{X_3(q)} = \langle D_Y \cup \{c\} \mid R_Y \cup R_3^c \rangle$  where  $D_Y = \{u, t, h\}$  (in the notations of Theorem 5.1 of [GKKaL3]) and  $R_3^c$  is a set of 5 relations involving  $c$  and elements of  $D_Y$ . Since  $Y \leq X_1(q)$ , obviously,  $D_Y \subseteq X_1(q)$ . Thus  $u, t$  and  $h$  can be expressed in terms of elements of  $D_1$ . Moreover, the relations  $R_Y$  hold, as they hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_Y$  and  $R_Y = R_{13}$ :

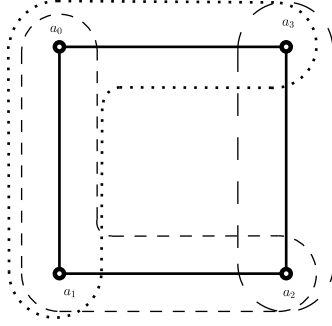
$$\sigma'_G = \langle D_1 \cup D_2 \cup \{c\} \mid R_1 \cup R_2 \cup R_3^c \cup R_{23} \rangle.$$

Notice that  $X_3(q) \cong \mathrm{SL}(3, q)$  is generated by its subgroups  $L_1$  and  $L_2$  (cf. Lemma 2.1 of [C2]). Hence,  $c$  can be expressed in terms of elements of  $D_1 \cup D_2$ . Finally, since  $\Delta_2 \cap \Delta_3 = A_1$  and  $A_1(q)$  is 2-generated (Proposition 2.6),  $|R_{23}| = 2$ . Therefore

$$\sigma_G^* = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^c \cup R_{23} \rangle,$$

and so  $G$  has a presentation with 5 generators and 26 relations.

4.3.  $\tilde{A}_3(q)$ . To obtain a presentation of  $G = \tilde{A}_3(q)$ , we start with Proposition 2.4.



Take  $\Delta_1 = A_3$  based on  $a_0, a_1$  and  $a_2$ ,  $\Delta_2 = A_3$  based on  $a_0, a_1$  and  $a_3$ , and  $\Delta_3 = A_2$  based on  $a_2$  and  $a_3$ . Thus  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \cup D_3 \mid R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup R_{23} \rangle.$$

Since  $X_i(q) \cong A_3(q)$ ,  $X_i(q) \cong \mathrm{SL}(4, q)$  by Proposition 2.3 for  $i = 1, 2$ . Now  $X_1(q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_3$  with 5 generators and 20 relations.

Consider a subgroup  $X$  of  $G$  generated by  $L_0$  and  $L_1$ . Its Dynkin diagram is of type  $A_2$  and so by Proposition 2.3,  $X \cong \mathrm{SL}(3, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_2$  with 4 generators and 14 relations. Now  $X \leq X_i(q)$  for  $i = 1, 2$ . Moreover  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_4$  with  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and relations  $R_X$  already hold. We use Tietze's transformations to eliminate  $D_X, R_X$  and  $R_{12}$ :

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \cup D_3 \mid R_1 \cup (R_2 \setminus R_X) \cup R_3 \cup R_{13} \cup R_{23} \rangle.$$

Since  $\sigma_4$  requires 6 generators and 22 relations,  $|D_1 \cup (D_2 \setminus D_X)| = 5 + 6 - 4 = 7$  and  $|R_1 \cup (R_2 \setminus R_X)| = 20 + 22 - 14 = 28$ .

Take now  $Y = L_3$ . Without loss of generality we may assume that  $\sigma_Y \subseteq \sigma_{X_3(q)}$  with  $\sigma_Y = \sigma_1$ . Then by Theorem 5.1 of [GKKaL3],  $\sigma_{X_3(q)} = \langle D_Y \cup \{c\} \mid R_Y \cup R_3^c \rangle$  where  $D_Y = \{u, t, h\}$  (in the notations of Theorem 5.1 of [GKKaL3]) and  $R_3^c$  is a set of 5 relations involving  $c$  and elements of  $D_Y$ . Since  $Y \leq X_2(q)$ , obviously,  $D_Y \subseteq X_2(q)$ . Thus  $u, t$  and  $h$  can be expressed in terms of elements of  $D_1 \cup (D_2 \setminus D_X)$ . Moreover, the relations  $R_Y$  hold, as they hold in  $X_2(q)$ . We use Tietze's transformations to eliminate  $D_Y$  and  $R_Y = R_{23}$ :

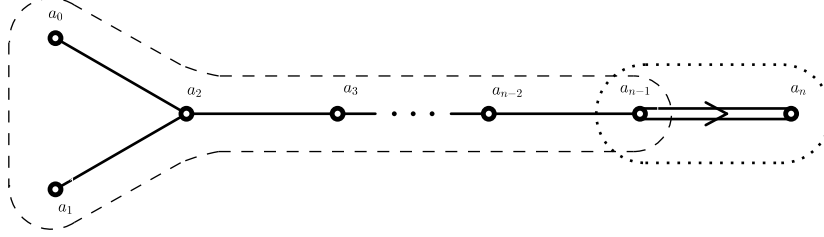
$$\sigma''_G = \langle D_1 \cup (D_2 \setminus D_X) \cup \{c\} \mid R_1 \cup (R_2 \setminus D_X) \cup R_3^c \cup R_{13} \rangle.$$

Notice that  $X_3(q) \cong \mathrm{SL}(3, q)$  is generated by its subgroups  $L_2$  and  $L_3$  (cf. Lemma 2.1 of [C2]). Hence,  $c$  can be expressed in terms of elements of  $D_1 \cup (D_2 \setminus D_X)$ . Finally, since  $\Delta_1 \cap \Delta_3 = A_1$  and  $A_1(q)$  is 2-generated (Proposition 2.6),  $|R_{13}| = 2$ . Therefore

$$\sigma_G^* = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus D_X) \cup R_3^c \cup R_{13} \rangle,$$

and so  $G$  has a presentation with 7 generators and 35 relations.

4.4.  $\tilde{B}_n(q)$ ,  $n \geq 9$ . To obtain a presentation of  $G = \tilde{B}_n(q)$ ,  $n \geq 4$ , we use Corollary 2.5.



By Proposition 2.3, the groups  $X_1(q)$ ,  $X_2(q)$  and  $X_3(q)$  are simply connected. Let  $\Delta_1$  be the subdiagram of  $\Delta$  whose vertices are the  $n$  nodes  $a_0, a_1, \dots, a_{n-1}$ . It has type  $D_n$  and so  $X_1(q) \cong \text{Spin}(2n, q)$ . Let  $\Delta_2$  be the subdiagram of  $\Delta$  whose vertices are the nodes  $a_{n-1}$  and  $a_n$ . It is of type  $C_2$  and so  $X_2(q) \cong \text{Sp}(4, q)$ . Then  $\Delta_3$  is the subdiagram of  $\Delta$  based on all vertices but  $a_{n-1}$ , thus of type  $D_{n-1} \times A_1$ . Hence,  $X_3(q) \cong \text{Spin}(2n-2, q) \times \text{SL}(2, q)$ . Clearly,  $\Delta = \cup_{i=1}^2 \Delta_i$ . Therefore  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

as described in Corollary 2.5.

Consider a subgroup  $X = L_{n-1}$  of  $G$ . Its Dynkin diagram is of type  $A_1$  and so by Proposition 2.3,  $X \cong \text{SL}(2, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$  with  $|D_X| = 3$  and  $|R_X| = 9$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ . The group  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even and  $\sigma_{X_2(q)} = \sigma_9$  if  $q$  is odd. By the results of Section 3,  $X_1(q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_{16}$  and  $\sigma_X \subseteq \sigma_{X_1(q)}$ .

Since  $X \leq X_2(q)$ , obviously,  $D_X \subseteq D_{X_2(q)}$  and the relations  $R_X$  hold, as they hold in  $X_2(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain:

$$\sigma'_G = \langle (D_1 \setminus D_X) \cup D_2 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3^* \rangle.$$

Now  $\sigma_{16}$  requires 7 generators and 34 relations, so that  $|(D_1 \setminus D_X) \cup D_2| = (7-3) + 5 = 9$  and  $|R_1 \cup (R_2 \setminus R_X)| = 34 + 27 - 9 = 52$  if  $q$  is odd, and  $|(D_1 \setminus D_X) \cup D_2| = (7-3) + 6 = 10$  and  $|R_1 \cup (R_2 \setminus R_X)| = 34 + 20 - 9 = 45$  if  $q$  is even.

Consider  $X_3(q) \cong \text{Spin}(2n-2, q) \times \text{SL}(2, q)$ . Each factor has two generators (Proposition 2.6). Denote them by  $a_1, a_2$  and  $b_1, b_2$  correspondingly. Then  $R_3^* = \{[a_i, b_j] = 1, 1 \leq i, j \leq 2\}$  and so  $|R_3^*| = 4$ .

Therefore  $G$  has a presentation with 10 generators, and 49 relations if  $q$  is even, and 9 generators and 56 relations if  $q$  is odd.

4.5.  $\tilde{B}_n(q)$ ,  $3 \leq n \leq 8$ . To obtain a presentation of  $G = \tilde{B}_3(q)$ , we use Corollary 2.5 similar to the case  $\tilde{B}_n(q)$  with  $n \geq 9$ . However, in this case we take  $\Delta_1 = A_3$  and  $\Delta_2 = C_2$ . Then  $X_1(q) \cong \text{SL}(4, q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_3$ , and  $X_2(q) \cong \text{Sp}_4(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_{18}$  if  $q$  is odd and  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even. Taking  $X = L_2$  (as in the previous case), we have that  $X \leq X_i(q)$  for  $i = 1, 2$ ,  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$  and  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq D_{X_1(q)}$  and  $R_X$  hold, as they hold in  $X_1(q)$ . Finally,  $X_3(q) = (X_1(q) \cap X_3(q)) \times (X_2(q) \cap X_3(q)) = (\text{SL}(2, q) \times \text{SL}(2, q)) \times \text{SL}(2, q)$ , and as before (using Proposition 2.6) we obtain that  $|R_3^*| = 4$ . Thus using Tietze's transformation we obtain that  $G$  has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle$$

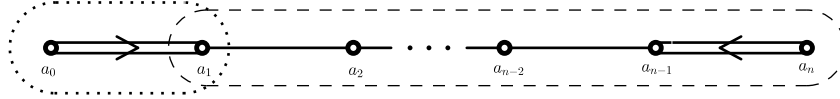
with  $5 + (6-3) = 8$  generators and  $20 + 28 - 9 + 4 = 43$  relations if  $q$  is odd, and generators and  $20 + 20 - 9 + 4 = 35$  relations of  $q$  is even.

To obtain a presentation of  $G = \tilde{B}_n(q)$  for  $4 \leq n \leq 8$ , we use Corollary 2.5 just as in the case  $\tilde{B}_n(q)$  with  $n \geq 9$ . If  $G = \tilde{B}_4(q)$ ,  $\Delta_1$  is of type  $D_4$ , and so  $X_1(q) \cong \text{Spin}(8, q)$  and has a presentation  $\sigma_{X_2(q)} = \sigma_{14}$  with 6 generators and 29 relations. Thus  $G$  has a presentation with

$(6 - 3) + 5 = 8$  generators and  $29 + 27 - 9 + 4 = 51$  relations if  $q$  is odd, and  $(6 - 3) + 6 = 9$  generators and  $29 + 20 - 9 + 4 = 44$  relations if  $q$  is even.

Finally, if  $G = \tilde{B}_n(q)$  with  $5 \leq n \leq 8$ ,  $\Delta_1$  is of type  $D_n$ , and so  $X_1(q) \cong \text{Spin}(2n, q)$  and has a presentation  $\sigma_{X_2(q)} = \sigma_{15}$  with 6 generators and 30 relations. Thus  $G$  has a presentation with  $(6 - 3) + 5 = 8$  generators and  $30 + 27 - 9 + 4 = 52$  relations if  $q$  is odd, and  $(6 - 3) + 6 = 9$  generators and  $30 + 20 - 9 + 4 = 45$  relations if  $q$  is even.

4.6.  $\tilde{C}_n(q)$ ,  $n \geq 3$ . To obtain a presentation of  $G = \tilde{C}_n(q)$ ,  $n \geq 3$ , we use Corollary 2.5.



By Proposition 2.3, the groups  $X_1(q)$ ,  $X_2(q)$  and  $X_3(q)$  are simply connected. Let  $\Delta_1$  be the subdiagram of  $\Delta$  whose vertices are the  $n$  nodes  $a_1, \dots, a_n$ . It has type  $C_n$  and so  $X_1(q) \cong \text{Sp}(2n, q)$ . Let  $\Delta_2$  be the subdiagram of  $\Delta$  whose vertices are the nodes  $a_0$  and  $a_1$ . It is of type  $C_2$  and so  $X_2(q) \cong \text{Sp}(4, q)$ . Then  $\Delta_3$  is the subdiagram of  $\Delta$  based on all vertices but  $a_1$ , thus of type  $A_1 \times C_{n-1}$ . Hence,  $X_3(q) \cong \text{SL}(2, q) \times \text{Sp}(2n - 2, q)$ . Clearly,  $\Delta = \Delta_1 \cup \Delta_2$ . Therefore  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

as described in Corollary 2.5.

Take a presentation  $\sigma_{X_1(q)} = \sigma_{11}$  if  $q$  is odd and  $\sigma_{X_1(q)} = \sigma_{12}$  if  $q$  is even. Consider a subgroup  $X = L_1$  of  $G$ . Its Dynkin diagram is of type  $A_1$  and so by Proposition 2.3,  $X \cong \text{SL}(2, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$  with  $|D_X| = 3$  and  $|R_X| = 9$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ . The group  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_9$  if  $q$  is odd, and  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even. By Theorem 7.1 of [GKKaL3],  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ , obviously,  $D_X \subseteq X_1(q)$ . Thus elements of  $D_X$  can be expressed in terms of elements of  $D_1$ . Moreover, the relations  $R_X$  hold, as they hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain:

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

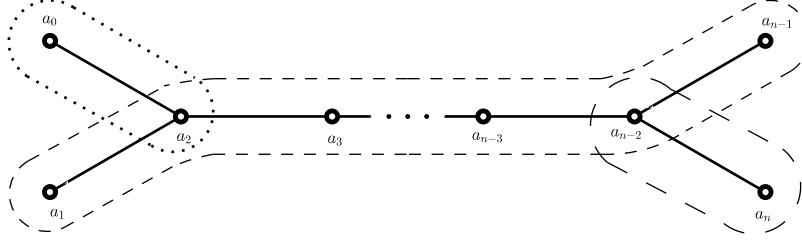
Therefore  $|D_1 \cup (D_2 \setminus D_X)| = 9 + 6 - 3 = 12$  and  $|R_1 \cup (R_2 \setminus R_X)| = 40 + 20 - 9 = 51$  if  $q$  is even, and  $|D_1 \cup (D_2 \setminus D_X)| = 8 + 5 - 3 = 10$  and  $|R_1 \cup (R_2 \setminus R_X)| = 47 + 27 - 9 = 65$  if  $q$  is odd.

Finally, consider  $X_3(q) \cong \text{Sp}(2n - 2, q) \times \text{SL}(2, q)$ . Each factor has two generators (Proposition 2.6). Thus as in the previous case we obtain  $|R_3^*| = 4$ .

Therefore  $G$  has a presentation with 12 generators and 55 relations if  $q$  is even, and 10 generators and 69 relations if  $q$  is odd.

4.7.  $\tilde{C}_2(q)$ . The only difference with the previous case is that  $\Delta_2 = C_2$ , and thus  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_9$  if  $q$  is odd and  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even. Replacing  $\sigma_{11}$  and  $\sigma_{12}$  by those, we obtain that  $G = \tilde{C}_2(q)$  has a presentation with  $6 + 6 - 3 = 9$  generators and  $20 + 20 - 9 + 4 = 35$  relations if  $q$  is even, and with  $5 + 5 - 3 = 7$  generators and  $27 + 27 - 9 + 4 = 49$  relations if  $q$  is odd.

4.8.  $\tilde{D}_n(q)$ ,  $n \geq 6$ . Let us assume that  $n \geq 9$ . We use Corollary 2.5. Let  $\Delta_1 = A_{n-1}$  on vertices  $a_1, a_2, \dots, a_{n-1}$ ,  $\Delta_2 = A_2 \times A_2$  on vertices  $a_0, a_2, a_{n-2}$  and  $a_n$ , and  $\Delta_3 = (A_1)^4 \times A_{n-5}$  on all vertices but  $a_2$  and  $a_{n-2}$ .



By Proposition 2.3,  $X_i(q)$  are simply connected for  $1 \leq i \leq 3$ , and so  $X_1(q) \cong \mathrm{SL}(n, q)$ ,  $X_2(q) \cong \mathrm{SL}(3, q) \times \mathrm{SL}(3, q)$  and  $X_3(q) \cong (\mathrm{SL}(2, q))^4 \times \mathrm{SL}(n-4, q)$ . Thus  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

Now  $X_1(q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_7$  with 6 generators and 25 relations.

Consider a subgroup  $X$  of  $G$  generated by  $L_2$  and  $L_{n-2}$ . Its Dynkin diagram is of type  $A_1 \times A_1$  and so by Proposition 2.3,  $X \cong \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$ . From Table 3 it follows that  $X$  has a presentation

$$\sigma_X = \langle D_X^2 \cup D_X^{n-2} \mid R_X^2 \cup R_X^{n-2} \cup R_X^* \rangle$$

where  $\sigma_X^i = \langle D_X^i \mid R_X^i \rangle$  is a presentation of  $L_i \cong \mathrm{SL}(2, q)$  for  $i = 2$  and  $n-2$ ,  $\sigma_X^i = \sigma_1$  and  $R_X^*$  are the relations that insure that  $[L_2, L_{n-2}] = 1$ . Since both  $L_2$  and  $L_{n-2}$  are 2-generated (Proposition 2.6), we may choose  $c_1, c_2 \in L_2$  and  $d_1, d_2 \in L_{n-2}$  so that  $R_X^* = \{[c_i, d_j] = 1, 1 \leq i, j \leq 2\}$ . Hence,  $|R_X^*| = 4$ .

Now  $X \leq X_i(q)$  for  $i = 1, 2$ . Moreover  $X_2(q)$  has a presentation  $\sigma_{X_2(q)}$  with  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Indeed, take  $\sigma_{X_2(q)} = \langle D_2 \mid R_2 \rangle$  such that  $D_2 = D_2^2 \cup D_2^{n-2}$  and  $R_2 = R_2^2 \cup R_2^{n-2} \cup R_2^*$  where for  $i = 2$  and  $n-2$ ,  $\sigma_{X_2(q)}^i = \langle D_2^i \mid R_2^i \rangle$  is a presentation of a subgroup  $L_{0,2}$  of  $X_2(q)$  if  $i = 2$ , and  $L_{n-2,n}$  if  $i = n-2$ ,  $\sigma_{X_2(q)}^i = \langle D_2^i \mid R_2^i \rangle = \sigma_2$  and  $R_2^*$  are the relations that insure that  $[L_{0,2}, L_{n-2,n}] = 1$ . Using [AsG] and [GK, p. 745, Corollary], we may take  $c'_1, c'_2 \in L_{0,2}$  and  $d'_1, d'_2 \in L_{n-2,n}$  with  $c'_1 = c_1$ ,  $d'_1 = d_1$ . Then let  $R_2^* = R_X^* \cup \{[c'_1, d'_2] = [c'_2, d'_1] = [c'_2, d'_2] = 1\}$ . Then  $R_X^* \subseteq R_2^*$  and  $|R_2^* \setminus R_X^*| = 3$ .

Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and relations  $R_X$  already hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$ :

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Notice that  $|D_1 \cup (D_2 \setminus D_X)| = 6 + (4-3) + (4-3) = 8$  and  $|R_1 \cup (R_2 \setminus R_X)| = 25 + (14-9) + (14-9) + 3 = 38$ .

Finally,  $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q))$  where  $X_3(q) \cap X_1(q) = L_1 \times \langle L_3, \dots, L_{n-3} \rangle \times L_{n-1} \cong \mathrm{SL}(2, q) \times \mathrm{SL}(n-4, q) \times \mathrm{SL}(2, q)$  and  $X_3(q) \cap X_2(q) = L_0 \times L_n \cong \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$ . Since each of the factors requires only two generators (Proposition 2.6), we obtain that  $|R_3^*| = 4$ . Therefore  $G$  has a presentation with 8 generators and 42 relations.

If  $6 \leq n \leq 8$ , then the above argument works with little variation. The subgroup  $X_1(q) \cong \mathrm{SL}(n, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_5$  with 5 generators and 21 relations. The rest of the argument does not change producing a presentation of  $G$  with 7 generators and 38 relations.

4.9.  $\tilde{D}_n(q)$ ,  $n = 4, 5$ . This time we use Corollary 2.5 with  $\Delta_1 = D_n$  based on all vertices but  $a_0$ , and  $\Delta_2 = A_2$  based on vertices  $a_0$  and  $a_2$ . Then  $X_1(q) \cong \mathrm{Spin}(2n, q)$  and  $X_2(q) \cong \mathrm{SL}(3, q)$ . Thus  $\Delta_3 = A_1^4$  if  $n = 4$ , and  $\Delta_3 = A_1^2 \times A_3$  if  $n = 5$ , giving  $X_3(q) \cong \mathrm{SL}(2, q)^4$  and  $X_3(q) \cong \mathrm{SL}(2, q)^2 \times \mathrm{SL}(4, q)$  correspondingly.

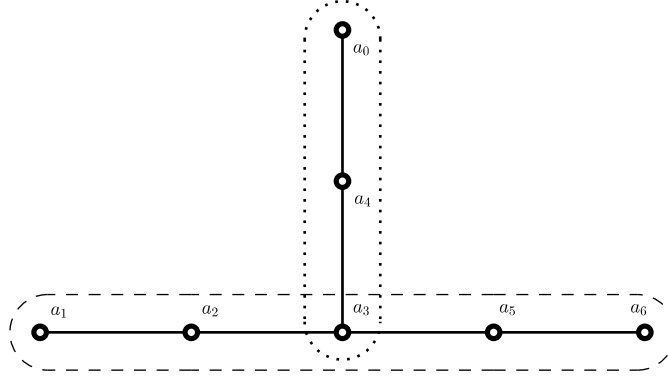
Consider a subgroup  $X = L_2$  of  $G$ . Then  $X \leq X_i(q)$  for  $i = 1, 2$ ,  $X \cong \mathrm{SL}(2, q)$  and  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ . Now  $X_1(q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_{14}$  if  $n = 4$  and  $\sigma_{X_1(q)} = \sigma_{15}$  if  $n = 5$ . The group  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_2$  and  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since

$X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and  $R_X$  hold as they hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain a presentation

$$\sigma_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle$$

where as usual  $|R_3^*| = 4$  (using Proposition 2.6). Thus  $G$  has a presentation with  $6 + 4 - 3 = 7$  generators and  $29 + (14 - 9) + 4 = 38$  relations if  $n = 4$ , and  $6 + 4 - 3 = 7$  generators and  $30 + (14 - 9) + 4 = 39$  relations if  $n = 5$ .

4.10.  $\tilde{E}_6(q)$ . This time we use Corollary 2.5. Take  $\Delta_1 = A_5$  on vertices  $a_1, a_2, a_3, a_5$  and  $a_6$ ,  $\Delta_2 = A_3$  on vertices  $a_0, a_4$  and  $a_3$ . Then  $\Delta_3 = A_2 \times A_2 \times A_2$  is based on all vertices but  $a_3$ .



Hence,  $G = \tilde{E}_6(q)$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

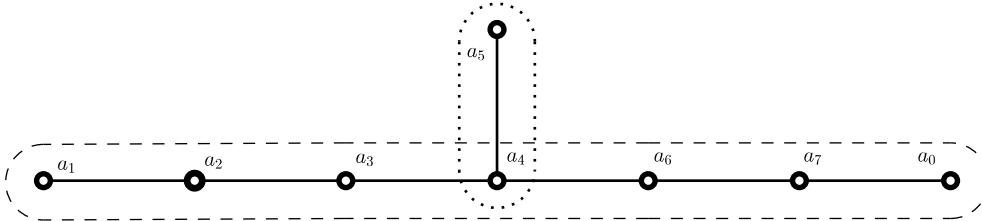
Using Proposition 2.3 and Table 3 we have that  $X_1 \cong \text{SL}(6, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_5$ , and  $X_2 \cong \text{SL}(4, q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_3$ .

Consider a subgroup  $X = L_3 \cong \text{SL}(2, q)$  of  $G$ . Then  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ , and Theorem 6.1 of [GKKaL3] implies that  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and the relations  $R_X$  hold (as they hold in  $X_1(q)$ ). We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$ . Hence,  $G$  has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally,  $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\text{SL}(3, q) \times \text{SL}(3, q)) \times \text{SL}(3, q)$ . Both factors require 2 generators (Proposition 2.6), implying  $|R_3^*| = 4$ . Therefore  $G$  has a presentation with  $5 + 5 - 3 = 7$  generators and  $21 + 20 - 9 + 4 = 36$  relations.

4.11.  $\tilde{E}_7(q)$ . Again we use Corollary 2.5. Take  $\Delta_1 = A_7$  based on all vertices but  $a_5$ , and  $\Delta_2 = A_3$  based on vertices  $a_4$  and  $a_5$ . Then  $\Delta_3 = A_3 \times A_3 \times A_1$  is based on all vertices but  $a_4$ .

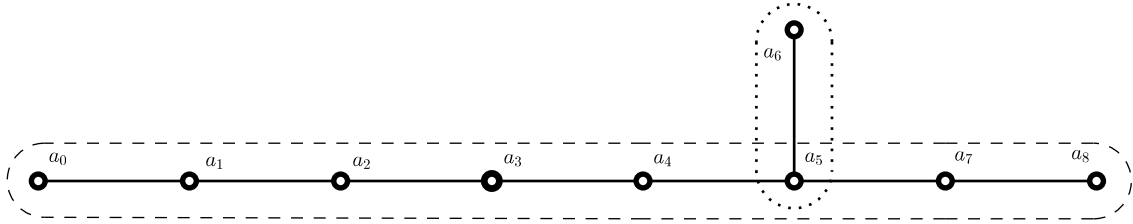


Hence,  $G = \tilde{E}_7(q)$  has a presentation as described in Corollary 2.5. Using Proposition 2.3 and Table 3 we have that  $X_1 \cong \text{SL}(8, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_5$ ,  $X_2 \cong \text{SL}(3, q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_2$ . Consider a subgroup  $X = L_4 \cong \text{SL}(2, q)$  of  $G$ . Then  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ , and  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and the relations  $R_X$  hold (as they hold in  $X_1(q)$ ). We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$ . Hence,  $G$  has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally,  $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\text{SL}(4, q) \times \text{SL}(4, q)) \times \text{SL}(2, q)$ . Both factors require 2 generators (Proposition 2.6), implying  $|R_3^*| = 4$ . Therefore  $G$  has a presentation with  $5 + 4 - 3 = 6$  generators and  $21 + 14 - 9 + 4 = 30$  relations.

4.12.  $\tilde{E}_8(q)$ . We use Corollary 2.5. Take  $\Delta_1 = A_8$  based on vertices but  $a_6$ , and  $\Delta_2 = A_2$  based on vertices  $a_5$  and  $a_6$ . Then  $\Delta_3 = A_5 \times A_2 \times A_1$  is based on all vertices but  $a_5$ .

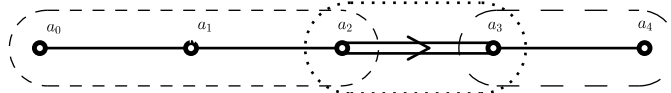


Hence,  $G = \tilde{E}_8(q)$  has a presentation as described in Corollary 2.5. Using Proposition 2.3 and Table 3 we have that  $X_1 \cong \text{SL}(9, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_7$ ,  $X_2 \cong \text{SL}(3, q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_2$ . Consider a subgroup  $X = L_5 \cong \text{SL}(2, q)$  of  $G$ . Then  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ , and  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ ,  $D_X \subseteq X_1(q)$  and the relations  $R_X$  hold (as they hold in  $X_1(q)$ ). We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$ . Hence,  $G$  has a presentation

$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally,  $X_3(q) = (X_3(q) \cap X_1(q)) \times (X_3(q) \cap X_2(q)) \cong (\text{SL}(6, q) \times \text{SL}(3, q)) \times \text{SL}(2, q)$ . Both factors require 2 generators (Proposition 2.6), implying  $|R_3^*| = 4$ . Therefore  $G$  has a presentation with  $6 + 4 - 3 = 7$  generators and  $25 + 14 - 9 + 4 = 34$  relations.

4.13.  $\tilde{F}_4(q)$ . This time we use Proposition 2.4 with  $k = 5$ . Take  $\Delta_1 = A_3$  based on vertices  $a_0, a_1$  and  $a_2$ ,  $\Delta_2 = C_2$  based on vertices  $a_2$  and  $a_3$ ,  $\Delta_3 = A_2$  based on vertices  $a_3$  and  $a_4$ ,  $\Delta_4 = A_2 \times A_2$  based on all vertices but  $a_2$ , and finally  $\Delta_5 = A_3 \times A_1$  based on all vertices but  $a_3$ .



Taking subgroups  $X_i(q)$  corresponding to  $\Delta_i$  for  $1 \leq i \leq 5$ , we obtain a presentation of  $G$

$$\sigma_G = \langle D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \mid R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup \bigcup_{i < j} R_{ij} \rangle$$

as described in Proposition 2.4. Notice that  $R_{13} = \emptyset$ .

Using Proposition 2.3 and Table 3 we have that  $X_1(q) \cong \mathrm{SL}(4, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_3$ ,  $X_2(q) \cong \mathrm{Sp}(4, q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_9$  if  $q$  is odd and  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even, and  $X_3(q) \cong \mathrm{SL}(3, q)$  has a presentation  $\sigma_{X_3(q)} = \sigma_2$ .

Take  $X = L_2 \cong \mathrm{SL}(2, q)$ . Then  $X \leq X_i(q)$  for  $i = 1, 2$  and Theorem 6.1 of [GKKaL3] implies that  $\sigma_X \subseteq \sigma_{X_1(q)}$ . Since  $X \leq X_2(q)$ ,  $D_X \subseteq X_2(q)$  and  $R_X$  hold as they hold in  $X_2(q)$ . We use Tietze's transformation to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain a presentation

$$\sigma_G^{(1)} = \langle (D_1 \setminus D_X) \cup D_2 \cup D_3 \cup D_4 \cup D_5 \mid (R_1 \setminus R_X) \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup \bigcup_{i < j} R_{ij} \setminus R_{12} \rangle.$$

Take  $Y = L_3 \cong \mathrm{SL}(2, q)$ . Then  $Y \leq X_i(q)$  for  $i = 2, 3$  and  $\sigma_Y \subseteq \sigma_{X_3(q)}$ . Again we use Tietze's transformation. This time we eliminate  $D_Y$ ,  $R_Y$  and  $R_{23}$  to obtain

$$\sigma_G^{(2)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \cup D_4 \cup D_5 \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4 \cup R_5 \cup \bigcup_{i < j} R_{ij} \setminus (R_{12} \cup R_{23}) \rangle.$$

By Proposition 2.3,  $X_4(q) = (X_4(q) \cap X_1(q)) \times X_3(q) \cong \mathrm{SL}(3, q) \times \mathrm{SL}(3, q)$ . Each factor is 2-generated (Proposition 2.6). Let us denote these pairs of generators by  $c_1, c_2$  and  $d_1, d_2$  correspondingly. In fact, [GK, p. 745, Corollary] allows us to choose  $c_1 \in L_0 \leq X_4(q) \cap X_1(q)$  and  $d_1 \in L_4 \leq X_3(q)$ . Then  $X_4(q)$  has a presentation  $\sigma_{X_4(q)} = \langle c_1, c_2, d_1, d_2 \mid R_{c_1, c_2} \cup R_{d_1, d_2} \cup R_4^* \rangle$  where  $\langle c_1, c_2 \mid R_{c_1, c_2} \rangle$  is a presentation of  $X_4(q) \cap X_1(q) \cong \mathrm{SL}(3, q)$ ,  $\langle d_1, d_2 \mid R_{d_1, d_2} \rangle$  is a presentation of  $X_4(q) \cap X_3(q) = X_3(q) \cong \mathrm{SL}(3, q)$ , and  $R_4^* = \{[c_1, d_1] = [c_1, d_2] = [c_2, d_1] = [c_2, d_2] = 1\}$ . Since  $X_4(q) \cap X_1(q) \leq X_1(q)$ ,  $c_1, c_2 \in X_1(q)$  and relations  $R_{c_1, c_2}$  hold as they hold in  $X_1(q)$ . Similarly,  $d_1, d_2 \in X_3(q)$  and relations  $R_{d_1, d_2}$  hold as they hold in  $X_3(q)$ . We now use Tietze's transformation to eliminate  $c_1, c_2, d_1, d_2$ ,  $R_{c_1, c_2} \cup R_{d_1, d_2}$  and  $R_{14} \cup R_{34}$ . Now we may eliminate relations  $R_{24}$ :  $R_{24}$  identifies  $X_2(q) \cap X_4(q)$ . Note that  $X_2(q) \cap X_4(q) = (X_2(q) \cap X_3(q)) \cap (X_3(q) \cap X_4(q))$ , and we have already identified  $X_2(q) \cap X_3(q)$  and  $X_3(q) \cap X_4(q)$ . Thus  $G$  has a presentation

$$\sigma_G^{(3)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \cup D_5 \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5 \cup \bigcup_{i=1}^5 R_{i5} \rangle.$$

Finally, by Proposition 2.3,  $X_5(q) = X_1(q) \times (X_3(q) \cap X_5(q)) \cong \mathrm{SL}(4, q) \times \mathrm{SL}(2, q)$ . Each factor is 2-generated (Proposition 2.6). Let us denote these pairs of generators by  $c'_1, c'_2$  and  $d'_1, d'_2$  correspondingly. Notice that [GK, p. 745, Corollary] implies that we may choose  $c'_1 = c_1$  and  $d'_1 = d_1$ . Then  $X_5(q)$  has a presentation  $\sigma_{X_5(q)} = \langle c'_1, c'_2, d'_1, d'_2 \mid R_{c'_1, c'_2} \cup R_{d'_1, d'_2} \cup R_5^* \rangle$  where  $\langle c'_1, c'_2 \mid R_{c'_1, c'_2} \rangle$  is a presentation of  $X_5(q) \cap X_1(q) = X_1(q) \cong \mathrm{SL}(4, q)$ ,  $\langle d'_1, d'_2 \mid R_{d'_1, d'_2} \rangle$  is a presentation of  $X_3(q) \cap X_5(q) \cong \mathrm{SL}(2, q)$ , and  $R_5^* = \{[c'_1, d'_1] = [c'_1, d'_2] = [c'_2, d'_1] = [c'_2, d'_2] = 1\}$ . Since  $X_5(q) \cap X_1(q) = X_1(q)$ ,  $c'_1, c'_2 \in X_1(q)$  and relations  $R_{c'_1, c'_2}$  hold as they hold in  $X_1(q)$ . Similarly,  $d'_1, d'_2 \in X_3(q)$  and relations  $R_{d'_1, d'_2}$  hold as they hold in  $X_3(q)$ . We use Tietze's transformation to eliminate  $c'_1, c'_2, d'_1, d'_2$ ,  $R_{c'_1, c'_2} \cup R_{d'_1, d'_2}$  and  $R_{15} \cup R_{35}$  to obtain

$$\sigma_G^{(4)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5^* \cup R_{25} \cup R_{45} \rangle.$$

Now we may eliminate relations  $R_{25}$ :  $R_{25}$  identify  $X_2(q) \cap X_5(q)$ . Note that  $X_2(q) \cap X_5(q) = (X_1(q) \cap X_2(q)) \cap (X_1(q) \cap X_5(q))$ , and we have already identified  $X_1(q) \cap X_2(q)$  and  $X_1(q) \cap X_5(q)$ . We may also eliminate relations  $R_{45}$ . Relations  $R_{45}$  identify  $X_4(q) \cap X_5(q)$  which is a direct product of two components:  $L_{01} = (X_1(q) \cap X_4(q)) \cap (X_1(q) \cap X_5(q))$  and  $L_4 = (X_3(q) \cap X_4(q)) \cap (X_3(q) \cap X_5(q))$ , and we have already identified those.

Notice that  $|R_4^* \cap R_5^*| = 1$  and so  $|R_4^* \cup R_5^*| = 7$ . Thus we obtained a presentation

$$\sigma_G^{(5)} = \langle (D_1 \setminus D_X) \cup D_2 \cup (D_3 \setminus D_Y) \mid (R_1 \setminus R_X) \cup R_2 \cup (R_3 \setminus R_Y) \cup R_4^* \cup R_5^* \rangle$$

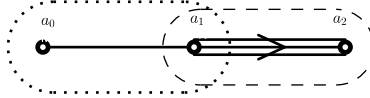
on  $5 - 3 + 6 + 4 - 3 = 9$  generators and  $(20 - 9) + 20 + (14 - 9) + 7 = 43$  relations if  $q$  is even, and  $5 - 3 + 5 + 4 - 3 = 8$  generators and  $(20 - 9) + 27 + (14 - 9) + 7 = 50$  relations if  $q$  is odd.



4.14.  $\tilde{G}_2(q)$ . We now use Corollary 2.5. Take  $\Delta_1$  based on  $a_0$  and  $a_1$ ,  $\Delta_2$  based on  $a_1$  and  $a_2$ , and  $\Delta_3$  based on  $a_0$  and  $a_1$ . Then  $X_1(q) \cong \text{SL}(3, q)$  has a presentation  $\sigma_{X_1(q)} = \sigma_2$ ,  $X_2(q) \cong G_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_{17}$ . Take  $X = L_1 \cong \text{SL}(2, q)$ . Then  $X \leq X_i(q)$  for  $i = 1, 2$ . Since  $X \leq X_2(q)$  and  $\sigma_X \subseteq \sigma_{X_1(q)}$ , we may use Tietze's transformation to remove  $D_X$ ,  $R_X$  and  $R_{12}$ , thus obtaining

$$\sigma_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

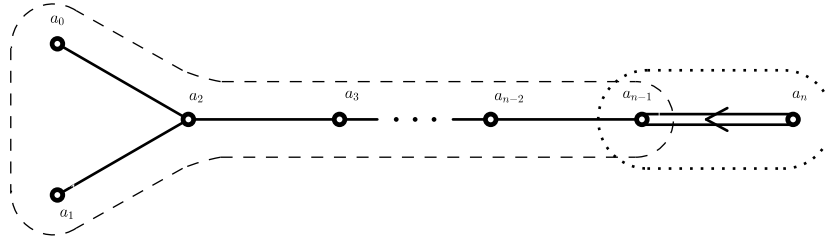
Since  $X_3(q) \cong \text{SL}(2, q) \times \text{SL}(2, q)$  and  $\text{SL}(2, q)$  is 2-generated (Proposition 2.6), we obtain  $|R_3^*| = 4$ , and so  $\sigma_G$  has  $4 + 6 - 3 = 7$  generators and  $14 + 31 - 9 + 4 = 40$  relations.



## 5. TWISTED AFFINE KAC-MOODY GROUPS

We briefly go through the calculations for the five remaining types of twisted affine Kac-Moody groups.

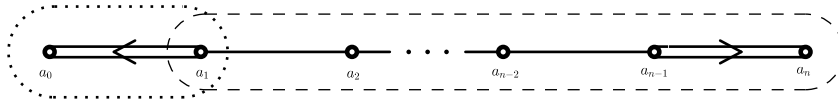
### 5.1. $\tilde{B}_n^t(q)$ .



For  $n \neq 3$ , the proof is line by line repetition of the case  $G = \tilde{B}_n(q)$  giving us the same result: a presentation of  $G$  with 9 generators and 56 relations when  $q$  is odd, and with 10 generators and 49 relations when  $q$  is even, for  $n \geq 9$ , and the same results as for  $\tilde{B}_n(q)$  for  $4 \leq n \leq 8$ .

Now for  $G = \tilde{B}_3^t(q)$ , we repeat the proof of the case  $\tilde{B}_3(q)$  one by line with one change: in the case when  $q$  is odd, we take  $\sigma_{X_2(q)} = \sigma_9$ , thus obtaining a presentation of  $G$  with  $5 + 5 - 3 = 7$  generators and  $20 + 27 - 9 + 4 = 42$  relations if  $q$  is odd, and 8 generators and 35 relations if  $q$  is even.

### 5.2. $\tilde{C}_n^t(q)$ . To obtain a presentation of $G = \tilde{C}_n^t(q)$ , $n \geq 3$ , we use Corollary 2.5.



By Proposition 2.3, the groups  $X_1(q)$ ,  $X_2(q)$  and  $X_3(q)$  are simply connected. Let  $\Delta_1 = B_n$  be the subdiagram of  $\Delta$  whose vertices are the  $n$  nodes  $a_1, \dots, a_n$ , and  $\Delta_2 = C_2$  the subdiagram of  $\Delta$  whose vertices are the nodes  $a_0$  and  $a_1$ . Then  $X_1(q) \cong \text{Spin}(2n + 1, q)$  and  $X_2(q) \cong \text{Sp}(4, q)$ .

It follows that  $\Delta_3$  is the subdiagram of  $\Delta$  based on all vertices but  $a_1$ , thus of type  $A_1 \times B_{n-1}$ . Hence,  $X_3(q) \cong \mathrm{SL}(2, q) \times \mathrm{Spin}(2n-1, q)$ . Clearly,  $\Delta = \Delta_1 \cup \Delta_2$ . Therefore  $G$  has a presentation

$$\sigma_G = \langle D_1 \cup D_2 \mid R_1 \cup R_2 \cup R_3^* \cup R_{12} \rangle.$$

as described in Corollary 2.5.

Take a presentation  $\sigma_{X_1(q)} = \sigma_{13}$  if  $q$  is odd and  $\sigma_{X_1(q)} = \sigma_{12}$  if  $q$  is even (notice that  $B_m(2^a) \cong C_m(2^a)$ ). Consider a subgroup  $X = L_1$  of  $G$ . Its Dynkin diagram is of type  $A_1$  and so by Proposition 2.3,  $X \cong \mathrm{SL}(2, q)$ . From Table 3 we know that  $X$  has a presentation  $\sigma_X = \langle D_X \mid R_X \rangle = \sigma_1$  with  $|D_X| = 3$  and  $|R_X| = 9$ . Now  $X \leq X_i(q)$  for  $i = 1, 2$ . The group  $X_2(q)$  has a presentation  $\sigma_{X_2(q)} = \sigma_{18}$  if  $q$  is odd, and  $\sigma_{X_2(q)} = \sigma_{10}$  if  $q$  is even. By Theorem 7.1 of [GKKaL3],  $\sigma_X \subseteq \sigma_{X_2(q)}$ . Since  $X \leq X_1(q)$ , obviously,  $D_X \subset X_1(q)$ . Thus elements of  $D_X$  can be expressed in terms of elements of  $D_1$ . Moreover, the relations  $R_X$  hold, as they hold in  $X_1(q)$ . We use Tietze's transformations to eliminate  $D_X$ ,  $R_X$  and  $R_{12}$  to obtain:

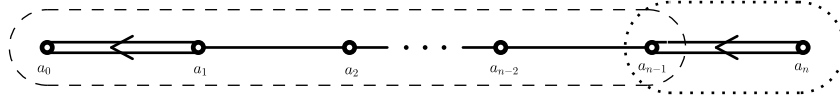
$$\sigma'_G = \langle D_1 \cup (D_2 \setminus D_X) \mid R_1 \cup (R_2 \setminus R_X) \cup R_3^* \rangle.$$

Finally, consider  $X_3(q) \cong \mathrm{Sp}(2n-2, q) \times \mathrm{SL}(2, q)$ . Each factor has two generators (Proposition 2.6). Thus as in the previous case we obtain  $|R_3^*| = 4$ .

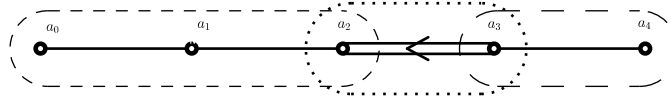
Therefore  $G$  has a presentation with  $9 + 6 - 3 = 12$  generators and  $48 + 28 - 9 + 4 = 71$  relations if  $q$  is odd, and  $9 + 6 - 3 = 12$  generators and  $40 + 20 - 9 + 4 = 55$  relations if  $q$  is even.

If  $n = 2$ , we use  $\sigma_{X_1(q)} = \sigma_9$  if  $q$  is odd and  $\sigma_{X_1(q)} = \sigma_{10}$  if  $q$  is even thus obtaining a presentation with  $5 + 6 - 3 = 8$  generators and  $27 + 28 - 9 + 4 = 50$  relations if  $q$  is odd, and  $6 + 6 - 3 = 9$  generators and  $20 + 20 - 9 + 4 = 35$  relations if  $q$  is even.

5.3.  $\tilde{C}'_n(q)$ . The proof is line by line repetition of the case  $G = \tilde{C}_n(q)$ . The outcome is the same: a presentation with 12 generators and 55 relations if  $q$  is even, and 10 generators and 69 relations if  $q$  is odd (with a smaller result for  $n = 2$ , cf. Table 1).

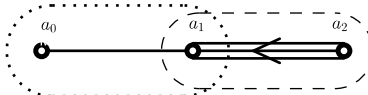


5.4.  $\tilde{F}_4^t(q)$ .



The proof is line by line repetition of the case  $G = \tilde{F}_4(q)$ . The outcome is the same: a presentation with 9 generators and 43 relations if  $q$  is even, and 8 generators and 50 relations if  $q$  is odd.

5.5.  $\tilde{G}_2^t(q)$ .



The argument follows the proof in the case  $G = \tilde{G}_2(q)$ . The outcome is the same: a presentation of  $G$  with 7 generators and 40 relations.

## 6. ADJOINT AND CLASSICAL GROUPS

So far we have worked with presentations of a simply connected Kac-Moody group  $X(q)$  defined over a finite field  $\mathbb{F} = \mathbb{F}_q$ . Our method can be used to derive a presentation of a Kac-Moody group that is not necessarily simply connected. In this section we deal with adjoint and classical groups. A reader can use our approach to derive a presentation of an arbitrary Kac-Moody group of his choosing.

There are two different meanings of the term *adjoint group*. For a group  $X(q)$ , its adjoint group is its image under the natural homomorphism  $X(q) \rightarrow \text{Aut}(X(q))$  given by the adjoint action:  $X(q)_{ad} := X(q)/Z(X(q))$ .

Besides the adjoint group  $X(q)_{ad}$ , there is also a group of points for an adjoint root datum. A convenient language to discuss this is a language of *group  $\mathbb{F}$ -functors*, the functors from the category of commutative  $\mathbb{F}$ -algebras to groups. A Kac-Moody group  $G_{\mathfrak{D}}(\mathbb{F})$  with a root datum  $\mathfrak{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$  is the result of applying a group functor  $G_{\mathfrak{D}}$  to the field  $\mathbb{F}$ . The Kac-Moody group  $G_{\mathfrak{D}}(\mathbb{F})$  is called *adjoint* if  $\Pi$  is a basis of  $\mathcal{X}$ . We denote the adjoint Kac-Moody group  $G_{\mathfrak{D}}(\mathbb{F})$  by  $X_{ad}(q)$ .

A homomorphism of the root data induces a homomorphism of the group  $\mathbb{F}$ -functors  $\pi : X \rightarrow X_{ad}$ . Taking points over  $\mathbb{F}$  yields a group homomorphism  $\pi(q) : X(q) \rightarrow X_{ad}(q)$ . The kernel of  $\pi(q)$  is the centre  $Z(X(q))$  of  $X(q)$ . Hence, we have an exact sequence of groups

$$1 \rightarrow Z(X(q)) \rightarrow X(q) \xrightarrow{\pi(q)} X_{ad}(q)$$

and  $X(q)_{ad}$  is observed in this sequence as the image of  $\pi(q)$ . For instance, if  $X = A_{n-1}$ , it is

$$1 \rightarrow \mu_n(\mathbb{F}_q) \rightarrow A_{n-1}(q) = \text{SL}_n(q) \xrightarrow{\pi(q)} (A_{n-1})_{ad}(q) = \text{PGL}_n(q)$$

where  $\mu_n$  is the group scheme of the  $n$ -th roots of unity and  $A_{n-1}(q)_{ad} = \text{PSL}_n(q)$  is the image of  $\pi(q)$  in  $\text{PGL}_n(q)$ .

Another insightful example is  $X = \tilde{A}_{n-1}$ . The key exact sequence is

$$1 \rightarrow \mu_n(\mathbb{F}_q) \times \mathbb{F}_q^\times \rightarrow \tilde{A}_{n-1}(q) = \tilde{\text{SL}}_n(\mathbb{F}_q[t, t^{-1}]) \xrightarrow{\pi(q)} (\tilde{A}_{n-1})_{ad}(q) = \mathbb{F}_q^\times \ltimes \text{PGL}_n(\mathbb{F}_q[t, t^{-1}])$$

where the simply connected group  $\tilde{A}_{n-1}(q)$  is the Steinberg central extension of  $\text{SL}_n(\mathbb{F}_q[t, t^{-1}])$  by  $\mathbb{F}_q^\times$  and the adjoint group  $(\tilde{A}_{n-1})_{ad}(q)$  is the semidirect product where the action of  $\mathbb{F}_q^\times$  is given by  $\alpha \cdot \sum_k P_k t^k = \sum_k \alpha^k P_k t^k$ .

Let  $\mathcal{P}$  be the weight lattice,  $\mathcal{Q}$  the root lattice of the corresponding Kac-Moody Lie algebra. The weight lattice  $\mathcal{P}$  is the root lattice  $\mathcal{X}$  for a simply connected root datum. *Mutatis mutandis*,  $\mathcal{Q}$  for an adjoint root datum. The natural map  $p : \mathcal{Q} \rightarrow \mathcal{P}$  is given by the Cartan matrix (or its transposed, depending on conventions). It is a part of an exact sequence

$$\mathcal{Q} \xrightarrow{p} \mathcal{P} \rightarrow \mathcal{Z} \rightarrow 1$$

where  $\mathcal{Z} = \text{coker } p$ . The Cartan matrix pinpoints all the tori (of the corresponding Kac-Moody groups) of interest for us:

$$Z(q) = Z(X(q)) = \text{hom}(\mathcal{Z}, \mathbb{F}_q^\times), \quad T(q) = \text{hom}(\mathcal{P}, \mathbb{F}_q^\times), \quad T_{ad}(q) = \text{hom}(\mathcal{Q}, \mathbb{F}_q^\times), \quad \pi(q)(\mathbf{x}) = \mathbf{x} \circ p.$$

Let us examine the corresponding (not exact) sequence of tori

$$1 \rightarrow Z(X(q)) \rightarrow T(q) \xrightarrow{\pi(q)} \overline{T}(q) \hookrightarrow T_{ad}(q)$$

where  $\overline{T}(q) = T(q)/Z(X(q))$  can be thought of as a torus of  $X(q)_{ad}$ .

**Proposition 6.1.** *Let  $X(q)$  be a simply-connected irreducible Kac-Moody group over a finite field  $\mathbb{F} = \mathbb{F}_q$  (finite, affine or indefinite). Let  $H(q) := H^1(\mathcal{Z}, \mathbb{F}_q^\times)$  in the finite or indefinite case and  $H(q) := H^1(\mathcal{Z}, \mathbb{F}_q^\times) \times \mathbb{F}_q^\times$  in the affine case. Then there exists a short exact sequence*

$$(1) \quad 1 \rightarrow X(q)_{ad} \rightarrow X_{ad}(q) \rightarrow H(q) \rightarrow 1.$$

*Proof.* Let us assume that  $X$  is of finite or indefinite type. Then  $p : \mathcal{Q} \rightarrow \mathcal{P}$  is injective and  $\mathcal{Z}$  is finite. The long exact sequence in cohomology

$$1 \rightarrow \text{hom}(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow \text{hom}(\mathcal{P}, \mathbb{F}_q^\times) \xrightarrow{\pi(q)} \text{hom}(\mathcal{Q}, \mathbb{F}_q^\times) \rightarrow H^1(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow 1$$

reduces to a short exact sequence connecting the adjoint tori

$$(2) \quad 1 \rightarrow \overline{T}(q) \rightarrow T_{ad}(q) \rightarrow H(q) \rightarrow 1.$$

This implies the existence of the short exact sequence (1).

If  $X$  is affine, the map  $p : \mathcal{Q} \rightarrow \mathcal{P}$  is no longer injective. We can decompose  $\mathcal{Q} = \mathcal{Q}' \times \mathbb{Z}$  where  $\mathbb{Z} = \ker p$  and  $p : \mathcal{Q}' \rightarrow \mathcal{P}$  is injective. The long exact sequence in cohomology is

$$1 \rightarrow \text{hom}(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow \text{hom}(\mathcal{P}, \mathbb{F}_q^\times) \xrightarrow{\pi(q)} \text{hom}(\mathcal{Q}', \mathbb{F}_q^\times) \rightarrow H^1(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow 1.$$

It gives a description of the tori using an auxiliary group  $T'(q) = \text{hom}(\mathcal{Q}', \mathbb{F}_q^\times)$ . The sequence

$$1 \rightarrow \overline{T}(q) \rightarrow T'(q) \rightarrow H^1(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow 1$$

is exact. Since  $T_{ad}(q) = T'(q) \times \mathbb{F}_q^\times$ , a direct product with  $\mathbb{F}_q^\times$  establishes exact sequence (2) in the affine case. This proves an existence of exact sequence (1) in all cases.  $\square$

Proposition 6.1 gives presentations of both  $X(q)_{ad}$  and  $X_{ad}(q)$ . Since  $X(q)_{ad} = X(q)/Z(q)$ , one gets  $X(q)_{ad}$  from  $X(q)$  by “killing” generators of  $Z(q)$ . The presentation of  $X_{ad}(q)$  is obtained from presentations of  $X(q)_{ad}$  and  $H(q)$  by the P. Hall’s Lemma [CLR, Lemma 2.2]. Observe that the right conjugations in the P. Hall’s Lemma are superfluous. One usually adds them for convenience.

**Corollary 6.2.** *Suppose we have a presentation of  $X(q)$ ,  $Z(q)$  and  $H(q)$ :*

$$\sigma_{X(q)} = \langle D \mid R \rangle, \quad \sigma_{Z(q)} = \langle D_1 \mid R_1 \rangle, \quad \sigma_{H(q)} = \langle D_2 \mid R_2 \rangle$$

*Then we have presentations of adjoint groups*

$$\sigma_{X(q)_{ad}} = \langle D \mid R \cup D_1^\sharp \rangle \text{ and } \sigma_{X_{ad}(q)} = \langle D \cup D_2 \mid R \cup D_1^\sharp \cup R_2^\sharp \cup D_2^{act} \rangle$$

where  $D_1^\sharp = \{x^\sharp = 1 \mid x \in D_1, x^\sharp \text{ is an expression of } x \text{ in } D\}$ ,

$R_2^\sharp = \{w = w^\sharp \mid w \in R_2, w^\sharp \in X(q)_{ad} \text{ is an expression of } w(D_2) \text{ in } D\}$  and

$D_2^{act} = \{xax^{-1} = {}^xa(D) \mid x \in D_2, a \text{ is a generator of } X(q)_{ad}, {}^xa(D) \text{ is an expression of } xax^{-1} \text{ in } D\}$ .

The group  $\mathcal{P}/\mathcal{Q}$  is computed by calculating the integral Smith normal forms of Cartan matrices. We summarise these calculations in Table 4. We list Dynkin’s labels for affine groups and Cartan labels for finite groups in the first column. The second column contains the group  $\mathcal{P}/\mathcal{Q}$ : by  $(a_1, \dots, a_k)$  we mean the group  $\mathbb{Z}/a_1 \times \dots \times \mathbb{Z}/a_k$ .

The next two columns are related to  $X(q)_{ad}$ . The third column lists the minimal number of generators for  $Z(q)$ . We get a generator, if there is a non-trivial homomorphism from a cyclic direct summand  $\mathbb{Z}/k$  of  $\mathcal{P}/\mathcal{Q}$  to  $\mathbb{F}_q^\times$ . We introduce a symbol  $\mathfrak{A}(k)$ , equal to 1, if  $\gcd(k, q-1) > 1$  and 0 if  $\gcd(k, q-1) = 1$ . The fourth column is a maximal possible value of  $\mathfrak{A}(k)$  taken over all  $q$ : this is the number of extra relations to describe  $X(q)_{ad}$  for generic  $q$ .

The right three columns are related to  $X_{ad}(q)$ . The fifth column lists the minimal number of generators for  $H(q)$ . We get a generator, if there is a non-trivial quotient by  $k$ -torsion,  $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)_k$ , where  $\mathbb{Z}/k$  is a direct summand of  $\mathcal{P}/\mathcal{Q}$ . We introduce a symbol  $\mathfrak{B}(k)$ , equal to 0 if  $q-1$  divides  $k$ , and 1 otherwise. The sixth column is a maximal possible value of  $\mathfrak{B}(k)$ : this is a number of extra generators needed to describe  $X_{ad}(q)$  for generic  $q$ . The last column is the maximal cardinality of  $D_1^\sharp \cup R_2^\sharp \cup D_2^{act}$ , the number of extra relations needed to describe  $X_{ad}(q)$ . In our computation we use the estimate  $|D_2^{act}| = 2|D_2|$  because  $X(q)$  is generated by 2 elements (with few exceptions, see Theorem 2.7)

As an application of our techniques we write down the numbers of generators and relations of remaining classical groups over  $\mathbb{F}_q[t, t^{-1}]$  in Table 5 (for sufficiently large  $q$ ). The groups  $\text{SL}_n$ ,  $\text{Spin}_n$  and  $\text{Sp}_{2n}$  are simply connected, so they are already in Tables 1 and 2. The group

$$\text{PSL}_n(\mathbb{F}_q[t, t^{-1}]) = \widetilde{A}_{n-1}(q)_{ad}$$

TABLE 4. Extra Generators and Relations for  $\tilde{X}(q)_{ad}$  and  $\tilde{X}_{ad}(q)$ 

$X_n$	$\mathcal{P}/\mathcal{Q}$	$ D_1 $	$\max  D_1 $	$ D_2 $	$\max  D_2 $	$\max  D_1^\# \cup R_2^\# \cup D_2^{act} $
$A_n$	$(n+1)$	$\mathfrak{A}(n+1)$	1	$\mathfrak{B}(n+1)$	1	4
$B_n, C_n, E_7$	(2)	$\mathfrak{A}(2)$	1	$\mathfrak{B}(2)$	1	4
$D_{2n}$	$(2, 2)$	$2\mathfrak{A}(2)$	2	$2\mathfrak{B}(2)$	2	9
$D_{2n+1}$	(4)	$\mathfrak{A}(2)$	1	$\mathfrak{B}(4)$	1	4
$G_2, F_4, E_8$	()	0	0	0	0	0
$E_6$	(3)	$\mathfrak{A}(3)$	1	$\mathfrak{B}(3)$	1	4
$\tilde{A}_{n-1}$	$(0, n)$	$1 + \mathfrak{A}(n)$	2	$1 + \mathfrak{B}(n)$	2	9
$\tilde{B}_n, \tilde{C}_n, \tilde{E}_7, \tilde{B}_n^t, \tilde{C}_n^t$	$(0, 2)$	$1 + \mathfrak{A}(2)$	2	$1 + \mathfrak{B}(2)$	2	9
$\tilde{D}_{2n}$	$(0, 2, 2)$	$1 + 2\mathfrak{A}(2)$	3	$1 + 2\mathfrak{B}(2)$	3	15
$\tilde{D}_{2n+1}$	$(0, 4)$	$1 + \mathfrak{A}(2)$	2	$1 + \mathfrak{B}(4)$	2	9
$\tilde{G}_2, \tilde{F}_4, \tilde{E}_8, \tilde{C}_n', \tilde{F}_4^t, \tilde{G}_2^t$	(0)	1	1	1	1	4
$\tilde{E}_6$	$(0, 3)$	$1 + \mathfrak{A}(3)$	2	$1 + \mathfrak{B}(3)$	2	9

is adjoint, hence its presentation follows from Tables 1 and 4. The groups

$$\mathrm{PGL}_n(\mathbb{F}_q[t, t^{-1}]) \triangleleft (\tilde{A}_{n-1})_{ad}(q), \quad \mathrm{SO}_{2n+1}(\mathbb{F}_q[t, t^{-1}]) \triangleleft (\tilde{B}_n)_{ad}(q)$$

are normal subgroups in the adjoint groups (before the semidirect product). Similarly to Proposition 6.1 they appear in an exact sequence

$$1 \rightarrow X(q)_{ad} \rightarrow \mathbf{G}(\mathbb{F}_q[t, t^{-1}]) \rightarrow H^1(\mathcal{Z}, \mathbb{F}_q^\times) \rightarrow 1,$$

hence they get a presentation as in Corollary 6.2 but with  $H^1(\mathcal{Z}, \mathbb{F}_q^\times)$  instead of  $H(q)$ .

Finally,  $\mathrm{SO}_{2n}$  is not related to the adjoint group. It is an intermediate quotient fitting into exact sequence of group schemes

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \mathrm{Spin}_{2n} \rightarrow \mathrm{SO}_{2n} \rightarrow 1.$$

Using our arguments, we fit the group into an exact sequence

$$1 \rightarrow \tilde{D}_{2n}(q)/Z \rightarrow \mathrm{SO}_{2n}(\mathbb{F}_q[t, t^{-1}]) \rightarrow H^1(\mathbb{Z}/2, \mathbb{F}_q^\times) \rightarrow 1$$

where the central subgroup  $Z$  is isomorphic to  $\mathrm{hom}(\mathbb{Z}/2, \mathbb{F}_q^\times)$ . We get a presentation as in Corollary 6.2 where the result depends on whether  $q$  is even or odd.

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TABLE 5. Generators and Relations of Classical  $\mathbf{G}(\mathbb{F}_q[t, t^{-1}])$ 

$\mathbf{G}$	$D_\sigma$	$R_\sigma$	$\mathbf{G}$	$D_\sigma$	$R_\sigma$	$D_\sigma$	$R_\sigma$
				$q$ odd		$q$ even	
$\mathrm{PSL}_3$	5	28	$\mathrm{SO}_7$	9	47	8	36
$\mathrm{PSL}_n, 4 \leq n \leq 8$	7	37	$\mathrm{SO}_9$	10	56	9	45
$\mathrm{PSL}_n, n \geq 9$	9	45	$\mathrm{SO}_{2n+1}, 5 \leq n \leq 8$	10	57	9	46
$\mathrm{PGL}_3$	6	31	$\mathrm{SO}_{2n+1}, n \geq 9$	11	61	10	50
$\mathrm{PGL}_n, 4 \leq n \leq 8$	8	40	$\mathrm{SO}_8$ or $\mathrm{SO}_{2n}, 6 \leq n \leq 8$	8	43	7	39
$\mathrm{PGL}_n, n \geq 9$	10	48	$\mathrm{SO}_{10}$	8	44	7	40
			$\mathrm{SO}_{2n}, n \geq 9$	9	47	8	43

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*E-mail address:* I.Capdeboscq@warwick.ac.uk, K.Kirkina@warwick.ac.uk, D.Rumynin@warwick.ac.uk